

STRICT SUPPORTS OF CANONICAL MEASURES AND APPLICATIONS TO THE GEOMETRIC BOGOMOLOV CONJECTURE

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ABSTRACT. The Bogomolov conjecture insists that a closed subvariety containing a dense subset of small points should be a kind of “special” subvarieties. In the arithmetic setting, the Bogomolov conjecture for abelian varieties has already been established as a theorem Ullmo and Zhang. However, it has not yet solved completely in the geometric setting, though there are some partial results.

In the study of the Bogomolov conjecture, the canonical measure is one of the key ingredients. The detailed investigation of this measure is crucially needed especially in the geometric setting. In this paper, we investigate the support of the canonical measure on a subvariety of an abelian variety, applying it to the geometric Bogomolov conjecture for abelian varieties. In fact, we show that the conjecture holds for abelian varieties satisfying some degeneration condition, which generalizes the result for totally degenerate abelian varieties due to Gubler and the recent work by the author. Further, we show that the conjecture in full generality holds true if the conjecture holds true for abelian varieties which are nowhere degenerate.

INTRODUCTION

0.1. Bogomolov conjecture and results. Let K be a number field, or a function field of normal projective variety over an algebraically closed base field k . We fix an algebraic closure \overline{K} of K . Let A be an abelian variety over \overline{K} and let L be an ample line bundle on A , and assume it is even, i.e., $[-1]^*L = L$. Then the canonical height function \hat{h}_L associated with L , also called the Néron-Tate height, is a semi-positive definite quadratic form on $A(\overline{K})$. It is well known that $\hat{h}_L(x) = 0$ if x is a torsion point. Let X be a closed subvariety of A . We put

$$X(\epsilon; L) := \left\{ x \in X(\overline{K}) \mid \hat{h}_L(x) \leq \epsilon \right\}$$

for a positive real number $\epsilon > 0$. Then the Bogomolov conjecture for abelian varieties, which is the main target in this paper, insists that $X(\epsilon; L)$ should not be Zariski-dense in X for some $\epsilon > 0$ unless X is something “special”, such as a torsion subvariety for example.

In the arithmetic case, the Bogomolov conjecture is now a theorem of Zhang and Ullmo:

Theorem A ([31] and [26], arithmetic version of Bogomolov conjecture). Let K be a number field. If X is not a torsion subvariety, then there is an $\epsilon > 0$ such that $X(\epsilon; L)$ is not Zariski dense in X .

In this paper, we consider the geometric version of this conjecture. In the geometric setting, the special subvarieties introduced in [29] are supposed to be counterparts to the

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torsion subvarieties in the arithmetic setting. Recall that X is called a *special subvariety* of A if there exist an abelian variety C over k , a closed subvariety $X' \subset C$, a homomorphism $\phi : C_{\overline{K}} \rightarrow A$, and a torsion point $\tau \in A(\overline{K})$ such that $X = G_X + \phi(X'_{\overline{K}}) + \tau$ (cf. Remark 5.1). It is known that, if X is a special subvariety, then $X(\epsilon; L)$ is Zariski dense in X for any $\epsilon > 0$ (cf. [29, Corollary 2.8]). The geometric Bogomolov conjecture insists that the converse should also hold true:

Conjecture B (Conjecture 2.9 in [29], Conjecture 5.2). If X is not a special subvariety, then there should exist $\epsilon > 0$ such that $X(\epsilon; L)$ is not Zariski dense in X .

We should give a remark on Moriwaki's result on an arithmetic version of the Bogomolov conjecture over a field K finitely generated over \mathbb{Q} . He constructed in [24] a kind of arithmetic height functions over K . This height is defined after a choice of arithmetic polarizations of K , and he established the arithmetic Bogomolov conjecture for abelian varieties with respect to the height associated with a *big* polarization. A classical geometric height is also one of his arithmetic heights, but his theory does not say anything about the geometric version of the conjecture because a geometric height does not arise from a big polarization.

Although the geometric version of the conjecture is still open, there are some partial answers to it. One important result is due to Gubler. He proved in [15] that the geometric Bogomolov conjecture holds for abelian varieties which are totally degenerate at some place. In this situation, the notion of special subvarieties coincides with that of torsion subvarieties. Recently, we proved in [29] a necessary condition for a subvariety to have dense small points.¹ Using Gubler's appendix in [29] moreover, we showed the following result, which is a generalization of the totally degenerate case:

Theorem C (Corollary 5.6 in [29]). Let K be a function field and let A be an abelian variety over \overline{K} . Suppose that there exists a place v such that $b(A_v) \leq 1$. Then the geometric Bogomolov conjecture holds for A .

Here A_v is the Berkovich analytic space associated to $A \times_{\text{Spec } \overline{K}} \text{Spec } \overline{K}_v$, where \overline{K}_v is the completion of \overline{K} with respect to a place v of \overline{K} (cf. § 5.1), and $b(A_v)$ denote the abelian rank of A_v (cf. § 1.4).

Main results in this paper also give partial answers to the geometric Bogomolov conjecture. For the statements, we would like to define the *nowhere-degeneracy rank* $\text{nd-rk}(A)$ of an abelian variety A over \overline{K} , where we let K be a function field. It is well known that for an abelian variety A , there exist simple abelian varieties A_1, \dots, A_r such that A is isogenous to $A_1 \times \dots \times A_r$. Renumbering them if necessary, we may assume that each A_i for $i = 1, \dots, s$ is degenerate at some place and each A_i for $i = s + 1, \dots, r$ is an abelian variety which has good reduction at any place.² We then define $\text{nd-rk}(A) := \dim(A_{s+1} \times \dots \times A_r)$, which can be checked to be well-defined for A (cf. Definition 5.7). Note that, for any $v \in M_{\overline{K}}$, we have $b(A_v) \geq \text{nd-rk } A$ and the equality holds if and only if all A_1, \dots, A_s are totally degenerate at v . The following statement is one of our main results in this paper, which generalizes Theorem C.

¹ We can show that the property “ $X(\epsilon; L)$ is dense in X for any $\epsilon > 0$ ” does not depend on an even ample line bundle L . We say X has dense small points if, for some (and hence any) L , $X(\epsilon; L)$ is dense for any $\epsilon > 0$ (cf. [29, Definition 2.2]).

²We may have $s = 0$.

Theorem D (Corollary 5.14). *Let A be an abelian variety over \overline{K} . Suppose that $\text{nd-rk } A \leq 1$. Then the geometric Bogomolov conjecture holds for A .*

Theorem D never insists that the geometric Bogomolov conjecture hold for *all* abelian varieties, but we can show that the conjecture for A can be reduced to the conjecture for $A_{s+1} \times \cdots \times A_r$ with the notation above. To be precise, we define in Definition 5.7 a notion of *nowhere degenerate factor* for A as the isogeny class of $A_{s+1} \times \cdots \times A_r$, which can be checked to be well-defined. Then we can show the following assertion:

Theorem E (Theorem 6.3). *Let A be an abelian variety and let B be a representative of the nowhere degenerate factor for A . Then the geometric Bogomolov conjecture holds for A if and only if it holds for B .*

As a corollary of Theorem E, we see that the geometric Bogomolov conjecture for abelian varieties is reduced to the conjecture for those without places of degeneration (cf. Conjecture 6.5). Note that Theorem E implies Theorem D because the geometric Bogomolov conjecture holds for all elliptic curves.

The Bogomolov conjecture for curves has been studied for a long time as one of the important special cases. In the arithmetic setting, Ullmo proved in [26] that it holds true. We consider the geometric version here. Then this conjecture insists that the embedded curve in its Jacobian should have only a finite number of small points unless it is isotrivial. See Conjecture 5.15 for the precise statement. There is also an effective version in Conjecture 5.16. Although these conjectures have not yet solved completely, some important results are established. In fact, Cinkir showed an affirmative answer to Conjecture 5.16 in [10] under the assumption that K is the function field of a curve over a field of characteristic zero. They are still open in positive characteristic, but there are some partial answers to Conjecture 5.16 such as in [21, 22] by Moriwaki and in [27, 28] by the author. We will give some remarks on the non-effective version Conjecture 5.15, which arises as consequences of our arguments.

0.2. Strategy and a problem. Our basic strategy is the same as that in [15] and [29]: to establish an argument analogous to the proof of Theorem A. We provide the reader with a brief summary of the proof of it first.

Step 1. Let L be an even ample line bundle on an abelian variety A over \mathbb{C} . It is well known that $[n]^*L \cong L^{\otimes n^2}$ and that there is a canonical metric h_{can} on L characterized by $[n]^*h_{can} = (h_{can})^{\otimes n^2}$, where $[n]$ is the multiplication endomorphism by n . Let X be a closed subvariety of A of dimension d . Taking the d -th wedge product of the curvature form $c_1(L, h_{can})$, dividing it by $\deg_L X$ and restricting to X , we obtain a probability measure

$$\mu_{X,L} = \frac{1}{\deg_L X} c_1(L, h_{can})^{\wedge d}|_X$$

on X . We call it the *canonical measure* on X with respect to L . We write simply μ_X for $\mu_{X,L}$ if we do not have to take care of L .

Step 2. Suppose that there exists a counterexample to Theorem A. Then, replacing K by a finite extension of it if necessary, we have an abelian variety A over a number field K and a geometrically irreducible closed subvariety $X \subset A$ which is not a torsion subvariety but has

dense small points. Taking the quotient by its stabilizer, we may further assume that X has trivial stabilizer. Consider a homomorphism

$$\alpha : A^N \rightarrow A^{N-1}, \quad (x_1, \dots, x_N) \mapsto (x_2 - x_1, \dots, x_N - x_{N-1}).$$

Since the stabilizer of X is trivial, this map restricts to a generically finite surjective morphism from $Z := X^N$ to its image $Y := \alpha(Z)$ for a natural number N large enough. We write the same symbol α for the restriction $Z \rightarrow Y$.

Since X has dense small points, we can take a generic sequence of small points of Z . Here, a sequence $(z)_n$ in $Z(\overline{K})$ is said to be *generic* if no subsequence of it is contained in a proper closed subset of Z , and is said to be *small* if $\lim_{n \rightarrow \infty} \hat{h}(z_n) = 0$, where \hat{h} is the canonical height associated to an even ample line bundle on Z .³ The image of it by α is also a generic sequence of small points of Y . By the equidistribution theorem, the $\text{Gal}(\overline{K}/K)$ -orbits of these two generic sequences are equidistributed with respect to the canonical measures μ_{Z_σ} and μ_{Y_σ} associated to even ample line bundles, where $\sigma : K \rightarrow \mathbb{C}$ is an infinite place and Z_σ and Y_σ are complex analytic spaces associated to Z and Y by σ . Since the generic sequence of Y is the image of a generic sequence of Z , we conclude $\alpha_* \mu_{Z_\sigma} = \mu_{Y_\sigma}$.

Step 3. On the other hand, μ_{Z_σ} and μ_{Y_σ} are measures defined by smooth and positive differential forms as we saw in Step 1. Then we can see that the equality $\alpha_* \mu_{Z_\sigma} = \mu_{Y_\sigma}$ cannot occur since the diagonal of $X_v^N = Z_v$ contracts to a point. Thus we obtain a contradiction, which proves Theorem A.

How can we make an analogy of the above argument then? Let K be a function field and let $\mathbb{K} = \overline{K}_v$ be the completion of \overline{K} with respect to a place v of \overline{K} (cf. § 5.1). Let A be an abelian variety over \mathbb{K} , $X \subset A$ a closed subvariety of dimension d and let L be an even ample line bundle on A . Suppose that all of them can be defined over \overline{K} . Then it is known that there also exists a canonical metric on L characterized by the same condition as the archimedean case. Moreover, we can define a Chambert-Loir measure $c_1(\overline{L}|_X)^{\wedge d}$. It is a semipositive Borel measure on the associated analytic space X^{an} , and hence the probability measure

$$\mu_{X^{\text{an}}, \overline{L}} := \frac{1}{\deg_L X} c_1(\overline{L}|_X)^{\wedge d}$$

on X^{an} is defined, also called a canonical measure. That is an analogy of Step 1.

An analogy of Step 2 can be obtained quite formally: Suppose that we have a counterexample to Conjecture B. Then we can find an abelian variety with a stabilizer-free closed subvariety X having dense small points. The morphism α and closed subvarieties Z and Y can be defined in the same way, and we can construct a generic net of small points in $Z(\overline{K})$.⁴ Let v be a place. Then the equidistribution theorem in [16] tells us $\alpha_* \mu_{Z_v} = \mu_{Y_v}$ as well.

Thus we have a good analogy up to here, but it is not a trivial task to deduce the contradiction from $\alpha_* \mu_{Z_v} = \mu_{Y_v}$ as we did in Step 3 above: It sometimes occurs that the support of μ_{Y_v} is just a single point for example, and then the equality $\alpha_*(\mu_{Z_v}) = \mu_{Y_v}$ does not lead us to a contradiction. This observation suggests that, if we try to establish some results concerning the geometric Bogomolov conjecture with an analogous method, some particular information on canonical measures depending on circumstances of the theorems

³The notion of “small” does not depend on the choice of even ample line bundles.

⁴A map from a directed set to a set S is called a *net* in S .

should be needed. In the setting of Theorem C in fact, we succeeded in finding a contradiction by focusing on the minimal dimension of the components of the support of the tropicalized canonical measure. The investigation of the canonical measures will also occupy an important position in this paper.

Let us give a remark on a limit of this strategy. As is mentioned in the previous subsection, Conjecture B can be reduced to Conjecture 6.5; the geometric Bogomolov conjecture for nowhere degenerate abelian varieties. The strategy described so far is useless against this conjecture, because the support of the canonical measure of a closed subvariety is necessarily a single point. This observation suggests that our theorems D and E are the ultimate which can be reached with this strategy based on the equidistribution theorems.

0.3. Outline of the proof. Our goals Theorem D and Theorem E are consequences of Theorem 5.4, which says that if X/G_X is tropically non-trivial (see Definition 5.3 for its definition), then X cannot have dense small points. The proof of it is delivered along the strategy explained above, and it works well by virtue of Theorem 4.5, which gives us a crucial information on the canonical measure. In this subsection, we describe what Theorem 4.5 says and give a sketch of the proof of Theorem 5.4 to see how it is used there.

Let A be an abelian variety over \mathbb{K} and let $X \subset A$ be a closed subvariety. Let \mathcal{X}' be a strictly semistable formal scheme with the Raynaud generic fiber X' and let $f : X' \rightarrow A^{\text{an}}$ be a generically finite morphism such that $f(X') = X^{\text{an}}$, with some technical assumptions. Let \overline{L} be an even ample line bundle on A with a canonical metric. Then we can define a probability measure $\mu_{X', f^* \overline{L}}$ on X' , which has a property that $f_* \mu_{X', f^* \overline{L}} = \mu_{X^{\text{an}}, L}$. Let $S(\mathcal{X}')$ be the skeleton of \mathcal{X}' , which is a simplicial set and a subspace of X' : We have a canonical simplex Δ_S for each stratum S of the special fiber of \mathcal{X}' and $S(\mathcal{X}') = \bigcup_S \Delta_S$. Gubler defined in [17] the notion of non-degenerate canonical simplices with respect to f , and showed that $\mu_{X', f^* \overline{L}}$ is a finite sum of the Lebesgue measures on the non-degenerate canonical simplices. It follows that the support $S_{X^{\text{an}}}$ of $\mu_{X^{\text{an}}, L}$ coincides with the image of the union of the non-degenerate canonical simplices by f . Moreover, he showed that $S_{X^{\text{an}}}$ has a unique piecewise affine structure such that the restriction of f to each non-degenerate canonical simplex is a piecewise affine map. We fix a sufficiently refined polytopal decomposition of $S_{X^{\text{an}}}$ here.⁵ A polytope σ in $S_{X^{\text{an}}}$ is called a *strict support* of $\mu_{X^{\text{an}}}$ if $\mu_{X^{\text{an}}} - \epsilon \delta_\sigma$ is semipositive for small $\epsilon > 0$ (cf. Definition 4.2). Suppose that σ is a strict support of $\mu_{X^{\text{an}}, L}$. It is not difficult to take a non-degenerate canonical simplex Δ_S with $\sigma \subset f(\Delta_S)$ and $\dim \sigma = \dim f(\Delta_S)$. Theorem 4.5 insist that any stratum Δ_S of $S(\mathcal{X}')$ with $\sigma \subset f(\Delta_S)$ and $\dim \sigma = \dim f(\Delta_S)$ should be a non-degenerate simplex with respect to f .

Let us give an outline of the proof of Theorem 5.4 now. We argue by contradiction, and suppose that it is not correct. Then we can take a stabilizer-free counterexample X such that S_{X_v} has positive dimension for some place v by the assumption that X/G_X is tropically non-trivial. We put $Z := X^N$ and consider $\alpha : Z \rightarrow Y$ as before. Taking into account that μ_{Z_v} can be regarded as the product of N copies of μ_{X_v} and that α contracts the diagonal of $Z = X^N$ to a point, we can find a strict support σ of μ_{Z_v} with $\dim \alpha(\sigma) < \dim \sigma$. Take a strictly semistable formal scheme \mathcal{Z}' with a generically finite surjective morphism $g : (\mathcal{Z}')^{\text{an}} \rightarrow Z_v$. Note that the morphism $h := \alpha \circ g$ as well as g has the same properties that f in the previous paragraph has because α is generically finite. Since σ is a strict

⁵It should be a descendible subdivisional one with the terminology in § 4.3.

support, there exists a non-degenerate canonical simplex Δ_S of $S(\mathcal{Z}')$ with respect to g such that $\sigma \subset g(\Delta_S)$ and $\dim \sigma = \dim g(\Delta_S)$. The inequality $\dim \alpha(\sigma) < \dim \sigma$ tells us $\dim \alpha(g(\Delta_S)) < \dim \Delta_S$, which implies that Δ_S is degenerate with respect to h . Moreover we have $\alpha(\sigma) \subset h(\Delta_S)$ and $\dim \alpha(\sigma) = \dim h(\Delta_S)$. Accordingly, we conclude that $\alpha(\sigma)$ is not a strict support of μ_{Y_v} by virtue of Theorem 4.5. On the other hand, since Z has dense small points, we can obtain $\alpha_*(\mu_{Z_v}) = \mu_{Y_v}$ by the equidistribution theorem. Since σ is a strict support of μ_{Z_v} , it can be seen that $\alpha(\sigma)$ is a strict support of μ_{Y_v} . Thus we obtain a contradiction, and hence Theorem 5.4 holds.

0.4. Organization. This article consists of six sections. In § 1, we recall some basic facts on non-archimedean geometry. The Raynaud extension of an abelian variety and its tropicalization recalled there will be important ingredients. § 2 is the most technical part in this paper. Its goal is to show that if there is a morphism from a strictly semistable formal scheme to an abelian variety, the induced morphism by reduction restricts to a torus-equivariant morphism between irreducible components. The goal of § 3 is Proposition 3.3. For a closed subvariety X of an abelian variety, we establish in this proposition a criterion when a stratum of the semistable alteration of a model of X is non-degenerate, in terms of the initial degeneration of X . It will be a key assertion to prove Theorem 4.5. We introduce the notion of strict support in § 4, and prove Theorem 4.5. The last two sections § 5 and § 6 are the climaxes of this paper. We will actually show that, our strategy, which is used in the proof of Theorem 5.4, works well by virtue of Theorem 4.5 in this situation. Using it, we give some partial answers to the geometric Bogomolov conjecture, including the main theorems mentioned in § 0.1. Some remarks on the conjecture for curves are also given in § 5.

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1. PRELIMINARY

We fix our conventions and terminology. When we write \mathbb{K} , it is an algebraically closed field which is complete with respect to a non-trivial non-archimedean absolute value $|\cdot| : \mathbb{K} \rightarrow \mathbb{R}$. We put $\mathbb{K}^\circ := \{a \in \mathbb{K} \mid |a| \leq 1\}$, the ring of integers of \mathbb{K} , $\mathbb{K}^{\circ\circ} := \{a \in \mathbb{K} \mid |a| < 1\}$, the maximal ideal of the valuation ring \mathbb{K}° , and further put $\tilde{\mathbb{K}} := \mathbb{K}^\circ / \mathbb{K}^{\circ\circ}$, the residue field. We put $\Gamma := \{-\log |a| \mid a \in \mathbb{K}^\times\}$, the value group of \mathbb{K} .

For an admissible formal scheme⁶ \mathcal{X} (cf. [14, 17]), we write $\tilde{\mathcal{X}} := \mathcal{X} \times_{\mathrm{Spf} \mathbb{K}^\circ} \mathrm{Spec} \tilde{\mathbb{K}}$. For a morphism $\varphi : \mathcal{X} \rightarrow \mathcal{Y}$ of admissible formal schemes, we write $\tilde{\varphi}$ for the induced morphism between their special fibers.

⁶When we say a formal scheme, it always means an admissible formal scheme in this article.

1.1. Berkovich analytic spaces. In this article, when we say an *analytic space*, it always means a Berkovich analytic space. We recall some notions and properties on analytic spaces associated to admissible formal schemes of algebraic varieties, as far as used later. For details, we refer to Berkovich's original papers [1, 2, 3, 4] or Gubler's expositions in his papers [14, 17].

Let \mathcal{X} be an admissible formal scheme over \mathbb{K}° . Then we can associate an analytic space \mathcal{X}^{an} , called the (*Raynaud*) *generic fiber* of \mathcal{X} . For a given analytic space X , an admissible formal scheme having X as the generic fiber is called a *formal model* of X . There is a reduction map $\text{red}_{\mathcal{X}} : \mathcal{X}^{\text{an}} \rightarrow \tilde{\mathcal{X}}$. Let Z be a dense open subset of an irreducible component of $\tilde{\mathcal{X}}$ with the generic point $\eta_Z \in Z$. Then there is a unique point $\xi_Z \in \mathcal{X}^{\text{an}}$ with $\text{red}_{\mathcal{X}}(\xi_Z) = \eta_Z$.

We can also associate an analytic space to an algebraic variety X over \mathbb{K} , and we write X^{an} for the analytic space associate to X . We have a natural inclusion $X(\mathbb{K}) \subset X^{\text{an}}$. We should recall the relationship between the analytic space associated to an algebraic variety and that done to an admissible formal scheme. Let \mathcal{X} be a scheme flat and of finite type over \mathbb{K}° with the generic fiber X . Let $\hat{\mathcal{X}}$ be the formal completion with respect to a nontrivial principal open ideal of \mathbb{K}° . Then it is an admissible formal scheme and $\hat{\mathcal{X}}^{\text{an}}$ is an analytic subdomain of X^{an} . In case that \mathcal{X} is proper over \mathbb{K}° , we have $\hat{\mathcal{X}}^{\text{an}} = X^{\text{an}}$.

Let Y be a closed subvariety of a proper algebraic variety X over \mathbb{K} and let \mathcal{X} be a formal model of X^{an} . Then there exists a unique admissible formal subscheme $\mathcal{Y} \subset \mathcal{X}$ with $\mathcal{Y}^{\text{an}} = Y^{\text{an}}$. We call this \mathcal{Y} the *closure* of Y in \mathcal{X} .

1.2. Tori. Let us fix the notations on the tori. Let \mathbb{G}_m^n denote the split torus of rank n over \mathbb{K} . In this section, let x_1, \dots, x_n denote the standard coordinates of \mathbb{G}_m^n unless otherwise noted, namely, $\mathbb{G}_m^n = \text{Spec } \mathbb{K}[(x_1)^\pm, \dots, (x_n)^\pm]$. Let $(\mathbb{G}_m^n)^{\text{an}}$ be the analytic space associated to \mathbb{G}_m^n . We put

$$(\mathbb{G}_m^n)_1^{\text{f-sch}} := \text{Spf } \mathbb{K}^\circ[(x_1)^\pm, \dots, (x_n)^\pm],$$

the formal torus over \mathbb{K}° , writing $(\mathbb{G}_m^n)_1^{\text{an}}$ for the generic fiber of $(\mathbb{G}_m^n)_1^{\text{f-sch}}$. We call $(\mathbb{G}_m^n)_1^{\text{f-sch}}$ the *canonical model* of $(\mathbb{G}_m^n)_1^{\text{an}}$. The analytic space $(\mathbb{G}_m^n)_1^{\text{an}}$ is an analytic subgroup of $(\mathbb{G}_m^n)^{\text{an}}$ as well as an affinoid subdomain of $(\mathbb{G}_m^n)^{\text{an}}$. The reduction $(\mathbb{G}_m^n)_{\mathbb{K}} := \text{Spec } \tilde{\mathbb{K}}[(x_1)^\pm, \dots, (x_n)^\pm]$ is called the *canonical reduction* of $(\mathbb{G}_m^n)_1^{\text{an}}$.

Each element $p \in (\mathbb{G}_m^n)^{\text{an}}$ can be regarded as a seminorm on $\mathbb{K}[(x_1)^\pm, \dots, (x_n)^\pm]$. We define a map $\text{val} : (\mathbb{G}_m^n)^{\text{an}} \rightarrow \mathbb{R}^n$, called the *valuation map*, by

$$p \mapsto (-\log p(x_1), \dots, -\log p(x_n)).$$

Note that $\text{val}^{-1}(\mathbf{0}) = (\mathbb{G}_m^n)_1^{\text{an}}$ (cf. [15, 4.3]).

1.3. Strictly semistable formal schemes, their skeletons, and subdivision. We first recall the notion of stratification of a variety Y over a field. We start with $Y^{(0)} := Y$. For each $r \in \mathbb{N}$, let $Y^{(r)} \subset Y^{(r-1)}$ be the complement of the set of normal points in $Y^{(r-1)}$. Since the set of normal points is open and dense, we obtain a chain of closed subsets:

$$Y = Y^{(0)} \supsetneq Y^{(1)} \supsetneq \dots \supsetneq Y^{(s)} \supsetneq Y^{(s+1)} = \emptyset.$$

The irreducible components of $Y^{(r)} \setminus Y^{(r-1)}$ are called the strata of Y , and the set of strata is denoted by $\text{str}(Y)$.

An admissible formal scheme \mathcal{X}' is called a *strictly semistable* formal scheme if any point of \mathcal{X}' has an open neighborhood \mathcal{U}' and an étale morphism

$$(1.0.1) \quad \psi : \mathcal{U}' \rightarrow \mathcal{S} := \mathrm{Spf} \mathbb{K}^\circ \langle x'_0, \dots, x'_d \rangle / (x'_0 \cdots x'_r - \pi),$$

where $\pi \in \mathbb{K}^\circ \setminus \{0\}$. Note that if we put $\mathcal{S}_1 := \mathrm{Spf} \mathbb{K}^\circ \langle x'_0, \dots, x'_r \rangle / (x'_0 \cdots x'_r - \pi)$ and $\mathcal{S}_2 := \mathrm{Spf} \mathbb{K}^\circ \langle x'_{r+1}, \dots, x'_d \rangle$, then we have $\mathcal{S} = \mathcal{S}_1 \times \mathcal{S}_2$. Let $\tilde{o} \in \tilde{\mathcal{S}}_1$ be the point defined by $x'_0 = \cdots = x'_r = 0$ in $\mathbb{K}[x'_0, \dots, x'_r] / (x'_0 \cdots x'_r)$. We recall here an assertion in [17]:

Proposition 1.1 (Proposition 5.2 in [17]). Any formal open covering of \mathcal{X}' admits a refinement $\{\mathcal{U}'\}$ by formal open subsets \mathcal{U}' as in (1.0.1) and which has the following properties:

- (a) Any \mathcal{U}' is a formal affine open subscheme of \mathcal{X}' .
- (b) There exists a distinguished stratum S of \mathcal{X}' associated to \mathcal{U}' such that for any stratum T of \mathcal{X}' , we have $S \subset \overline{T}$ if and only if $\tilde{\mathcal{U}}' \cap \overline{T} \neq \emptyset$, where \overline{T} is the closure of T in \mathcal{X}' .
- (c) The subset $\tilde{\psi}^{-1}(\{\tilde{o}\} \times \tilde{\mathcal{S}}_2)$ is the stratum of $\tilde{\mathcal{U}}'$ which is equal to $\tilde{\mathcal{U}}' \cap S$ for the distinguished stratum S from (b).
- (d) Any stratum of \mathcal{X}' is the distinguished stratum of a suitable \mathcal{U}' .

For a strictly semistable formal scheme \mathcal{X}' , we can define a subspace $S(\mathcal{X}')$ of $X' := (\mathcal{X}')^{\mathrm{an}}$, called the *skeleton*. It has a canonical structure of an abstract simplicial set which reflects the incidence relations between the strata of \mathcal{X}' . We briefly recall it here, and refer to [17, 5.3] for more details. For any stratum S of \mathcal{X}' of codimension r , take a formal affine open subset \mathcal{U}' of \mathcal{X}' such that S is the distinguished stratum of \mathcal{U}' , and an étale morphism as in (1.0.1). It is well known that the first projection $\mathcal{S} \rightarrow \mathcal{S}_1$ induces an isomorphism $S(\mathcal{S}) \cong S(\mathcal{S}_1)$ between the skeletons and we have an isomorphism

$$S(\mathcal{S}_1) \cong \{(u'_0, \dots, u'_r) \in \mathbb{R}_{\geq 0}^{r+1} \mid u'_0 + \cdots + u'_r = v(\pi)\} \cong \Delta',$$

where the second isomorphism is given by omitting u'_0 (cf. [17, 5.3]). On the other hand, the skeleton $S(\mathcal{U}')$ of $U' := (\mathcal{U}')^{\mathrm{an}}$ is a subset of $S(\mathcal{X}')$ and ψ in (1.0.1) induces an isomorphism $S(\mathcal{S}) \cong S(\mathcal{S}_1)$ between skeletons. Accordingly, we have

$$(1.1.2) \quad S(\mathcal{U}') \cong S(\mathcal{S}) \cong S(\mathcal{S}_1) \cong \Delta'$$

and, in particular, the subset $S(\mathcal{U}')$ of $S(\mathcal{X}')$ is a simplex. Since $S(\mathcal{U}')$ depends only on S , not on the choice of \mathcal{U}' , we write $\Delta_S = S(\mathcal{U}')$ and call it the *canonical simplex* corresponding to S . It is known that the canonical simplices $\{\Delta_S\}_{S \in \mathrm{Str}(\mathcal{X}')}$ cover $S(\mathcal{X}')$, which gives a canonical structure of an abstract simplicial set to the skeleton $S(\mathcal{X}')$.

There is a continuous map $\mathrm{Val} : X' \rightarrow S(\mathcal{X}')$ which restricts to the identity on $S(\mathcal{X}')$. If S is a distinguished stratum of \mathcal{X}' associated to \mathcal{U}' in the sense of Proposition 1.1, then the restriction of Val to $U' := (\mathcal{U}')^{\mathrm{an}}$ is described as follows: We can regard U' as a rational subdomain of $(\mathbb{G}_m^r)^{\mathrm{an}}$ with the standard coordinates x'_1, \dots, x'_r , by omitting x'_0 . Let $\mathrm{val}' : (\mathbb{G}_m^r)^{\mathrm{an}} \rightarrow \mathbb{R}^r$ be the valuation map as in § 1.2. Using the identification

$$(1.1.3) \quad \Delta_S \cong \Delta' := \{(u'_1, \dots, u'_r) \in \mathbb{R}_{\geq 0}^r \mid u'_1 + \cdots + u'_r \leq v(\pi)\}$$

given by (1.1.2), we can describe Val as $\mathrm{Val}(p) = \mathrm{val}'(\psi^{\mathrm{an}}(p)) \in \Delta' = \Delta_S$ for $p \in U'$. See [4] or [17, 5.3] for more details.

Let \mathcal{X}' be a strictly semistable formal scheme. Let \mathcal{D} be a Γ -rational subdivision of the skeleton $S(\mathcal{X}')$. This means that \mathcal{D} is a family of Γ -rational polytopes⁷, each contained in a canonical simplex, such that $\{\Delta \in \mathcal{D} \mid \Delta \subset \Delta_S\}$ is a polytopal decomposition of Δ_S for any stratum S of $\tilde{\mathcal{X}}'$ (cf. [17, 5.4]). Then [17, Proposition 5.5] gives us a unique formal scheme \mathcal{X}'' corresponding to \mathcal{D} . Furthermore, [17, Proposition 5.7] tells us that there is a bijective correspondence between the set of strata of $\tilde{\mathcal{X}}''$ and \mathcal{D} . More precisely:

Proposition 1.2 (Proposition 5.7 in [17]). Let \mathcal{X}'' be the formal scheme associated to \mathcal{D} as in [17, Proposition 5.5]. Then there exists a bijective correspondence between open faces τ of \mathcal{D} and strata R of $\tilde{\mathcal{X}}''$ given by

$$R = \text{red}_{\mathcal{X}''}(\text{Val}^{-1}(\tau)), \quad \tau = \text{Val}(\text{red}_{\mathcal{X}''}^{-1}(Y)),$$

where Y is any non-empty subset of R .

As a corollary of the proof of [17, Proposition 5.7], Gubler established the following assertion:

Proposition 1.3 (cf. Corollary 5.9 in [17]). Let \mathcal{X}'' be the formal scheme associated to \mathcal{D} as in [17, Proposition 5.5] and let $\iota' : \mathcal{X}'' \rightarrow \mathcal{X}'$ be the morphism extending the identity on $(\mathcal{X}')^{\text{an}}$. Let $u \in \mathcal{D}$ be a vertex and let R be the stratum of $\tilde{\mathcal{X}}''$ corresponding to u (cf. Proposition 1.2). Then $S := \tilde{\iota}'(R)$ is a stratum of $\tilde{\mathcal{X}}'$ with $u \in \text{relin } \Delta_S$. Furthermore, if we put $r := \dim R - \dim S$, then $\iota'|_R : R \rightarrow S$ has a structure of $(\mathbb{G}_m^r)_{\mathbb{K}}$ -torsor.

We will see in the proof of Lemma 2.1 how the $(\mathbb{G}_m^r)_{\mathbb{K}}$ -torsor in Proposition 1.3 can be described.

1.4. The Raynaud extension and the valuation map. We recall here some notions on the Raynaud extensions as far as needed in the sequel. See [7, §1] and [17, §4] for details.

Let A be an abelian variety over \mathbb{K} . According to [7, Theorem 1.1], there exists a unique analytic subgroup $A^\circ \subset A^{\text{an}}$ with a formal model \mathcal{A}° having the following properties:

- \mathcal{A}° is a formal group scheme and $(\mathcal{A}^\circ)^{\text{an}} \cong A^\circ$ as group analytic spaces.
- There is a short exact sequence

$$(1.3.4) \quad 1 \longrightarrow \mathcal{T}^\circ \longrightarrow \mathcal{A}^\circ \longrightarrow \mathcal{B} \longrightarrow 0,$$

where $\mathcal{T}^\circ \cong (\mathbb{G}_m^n)_1^{\text{f-sch}}$ for some $n \geq 0$, and \mathcal{B} is a formal abelian variety.

By virtue of [5, Satz 1.1], we see that such an \mathcal{A}° is unique, and \mathcal{T}° and \mathcal{B} are also uniquely determined. Taking the generic fiber of the above exact sequence, we obtain an exact sequence

$$(1.3.5) \quad 1 \longrightarrow T^\circ \longrightarrow A^\circ \xrightarrow{(q^\circ)^{\text{an}}} B \longrightarrow 0$$

of group spaces. We call \mathcal{T}° , \mathcal{A}° and \mathcal{B} the *canonical formal models* of T° , A° and B respectively.

Naturally T° is an analytic subgroup of the analytic torus $T \cong (\mathbb{G}_m^n)^{\text{an}}$. Pushing (1.3.5) out by $T^\circ \hookrightarrow T$, we obtain an exact sequence

$$(1.3.6) \quad 1 \longrightarrow T \longrightarrow E \xrightarrow{q^{\text{an}}} B \longrightarrow 0,$$

⁷ We adopt the notation and terminology in 6.1 and Appendix in [14]

which is called the *Raynaud extension* of A . The natural morphism $A^\circ \rightarrow E$ is an immersion of analytic groups. [7, Theorem 1.2] says that the homomorphism $T^\circ \hookrightarrow A^{\text{an}}$ extends uniquely to a homomorphism $T \rightarrow A^{\text{an}}$ and hence to a homomorphism $p^{\text{an}} : E \rightarrow A^{\text{an}}$. This p^{an} is also called the *Raynaud extension* of A by abuse of words. It is known that p^{an} is a surjective homomorphism and moreover $M := \text{Ker } p^{\text{an}}$ is a lattice in $E(\mathbb{K})$. Thus A^{an} can be described as a quotient of E by the lattice M .

The dimension of T is called the *torus rank* of A , and the dimension of B is called *abelian rank* of A . We denote the abelian rank of A by $b(A)$. A is said to be *degenerate* if $b(A) < \dim A$, or equivalently if the torus rank of A is positive. Note that “being non-degenerate” means “having good reduction”.

Taking into account the fact that the transition functions of the T -torsor (1.3.6) can be valued in T° , we can define a continuous map

$$(1.3.7) \quad \text{val} : E \rightarrow \mathbb{R}^n,$$

as in [7], where n is the torus rank of A . We recall here how it can be described. Let us first fix an isomorphism $T^\circ \cong (\mathbb{G}_m^n)_1^{\text{an}}$, with the standard coordinates x_1, \dots, x_n . We can take a covering $\{V\}$ of B consisting of rational subdomains and trivializations

$$(1.3.8) \quad ((q^\circ)^{\text{an}})^{-1}(V) \cong V \times (\mathbb{G}_m^n)_1^{\text{an}}$$

as $(\mathbb{G}_m^n)_1^{\text{an}}$ -torsors for all V . Since the Raynaud extension is the push-out of (1.3.5) by the canonical inclusion $(\mathbb{G}_m^n)_1^{\text{an}} \hookrightarrow (\mathbb{G}_m^n)^{\text{an}}$, the isomorphism (1.3.8) extend to isomorphisms

$$(1.3.9) \quad (q^{\text{an}})^{-1}(V) \cong V \times (\mathbb{G}_m^n)^{\text{an}}$$

of $(\mathbb{G}_m^n)^{\text{an}}$ -torsors, and we obtain morphisms

$$(1.3.10) \quad r_V : (q^{\text{an}})^{-1}(V) \cong V \times (\mathbb{G}_m^n)^{\text{an}} \rightarrow (\mathbb{G}_m^n)^{\text{an}}$$

for all V , by composing with the second projection. A different choice of (1.3.8) gives a different isomorphism in (1.3.9) and hence a different morphism in (1.3.10) for each V , but the difference is only the multiplication of an element of $(\mathbb{G}_m^n)_1^{\text{an}}$ on $(\mathbb{G}_m^n)^{\text{an}}$. Accordingly, the morphisms $(q^{\text{an}})^{-1}(V) \rightarrow \mathbb{R}^n$ given by

$$e \mapsto (-\log r_V(e)(x_1), \dots, -\log r_V(e)(x_n))$$

patch together to be a morphism from E to \mathbb{R}^n . It is our valuation map $\text{val} : E \rightarrow \mathbb{R}^n$.

The image $\Lambda := \text{val}(M) \subset \mathbb{R}^n$, also contained in Γ^n , is a complete lattice in \mathbb{R}^n and we have a diagram

$$\begin{array}{ccc} E & \xrightarrow{\text{val}} & \mathbb{R}^n \\ \downarrow & & \downarrow \\ A^{\text{an}} & \xrightarrow{\overline{\text{val}}} & \mathbb{R}^n / \Lambda \end{array}$$

that commutes. The homomorphism $\overline{\text{val}}$ is also called the valuation map. From the construction of val and $\overline{\text{val}}$, we see $A^\circ = \overline{\text{val}}^{-1}(\overline{\mathbf{0}}) \cong \text{val}^{-1}(\mathbf{0})$.

1.5. Homomorphism, products and Raynaud extensions. Let A_1 and A_2 be abelian varieties over \mathbb{K} and let $\phi : A_1 \rightarrow A_2$ be a homomorphism. Let

$$1 \longrightarrow T_i \longrightarrow E_i \xrightarrow{q_i^{\text{an}}} B_i \longrightarrow 0$$

be the Raynaud extension of A_i for $i = 1, 2$. Then [6, Proposition 2.2] tells us that ϕ induces a homomorphism $A_1^\circ \rightarrow A_2^\circ$ and hence we obtain a homomorphism $T_1^\circ \rightarrow T_2^\circ$. Accordingly we have an induced homomorphism $\phi_{ab} : B_1 \rightarrow B_2$ between the abelian parts. Furthermore, we can extend ϕ to a homomorphism between the Raynaud extensions. In fact, since the homomorphism $T_1^\circ \rightarrow T_2^\circ$ is algebraizable (cf. [12, Corollaire 3.4 and Théorème 3.6]), it canonically extends to $T_1 \rightarrow T_2$, and hence we obtain, from the construction of the Raynaud extensions, a homomorphism $\Phi : E_1 \rightarrow E_2$ which is a lift of ϕ^{an} . It is not difficult to see from the construction of the valuation map that Φ descends to a linear map $\phi_{\text{aff}} : \mathbb{R}^{n_1} \rightarrow \mathbb{R}^{n_2}$ by the valuation maps, where n_1 and n_2 are the torus rank of A_1 and A_2 respectively, and we have $\phi_{\text{aff}}(\Lambda_1) \subset \Lambda_2$. Therefore, we have a homomorphism $\bar{\phi}_{\text{aff}} : \mathbb{R}^{n_1}/\Lambda_1 \rightarrow \mathbb{R}^{n_2}/\Lambda_2$ which makes the diagram

$$(1.3.11) \quad \begin{array}{ccc} A_1^{\text{an}} & \xrightarrow{\phi_{\text{aff}}} & A_2^{\text{an}} \\ \downarrow & & \downarrow \\ \mathbb{R}^{n_1}/\Lambda_1 & \xrightarrow{\bar{\phi}_{\text{aff}}} & \mathbb{R}^{n_2}/\Lambda_2 \end{array}$$

commutative.

Next we consider the direct product. We put $A := A_1 \times A_2$. Then we have $A_1^\circ \times A_2^\circ \subset A^{\text{an}}$ and an exact sequence

$$1 \rightarrow T_1^\circ \times T_2^\circ \rightarrow A_1^\circ \times A_2^\circ \rightarrow B_1 \times B_2 \rightarrow 0.$$

We can see $A^\circ = A_1^\circ \times A_2^\circ$ and $T^\circ = T_1^\circ \times T_2^\circ$ from their definitions, and further we find

$$(1.3.12) \quad 1 \rightarrow T_1 \times T_2 \rightarrow E_1 \times E_2 \rightarrow B_1 \times B_2 \rightarrow 0$$

is the Raynaud extension of A . We can also see that

$$\overline{\text{val}} : (A_1 \times A_2)^{\text{an}} \rightarrow \mathbb{R}^{n_1+n_2}/(\Lambda_1 \oplus \Lambda_2)$$

coincides with the map between the products

$$A_1^{\text{an}} \times A_2^{\text{an}} \rightarrow \mathbb{R}^{n_1}/\Lambda_1 \times \mathbb{R}^{n_2}/\Lambda_2$$

induced from $\overline{\text{val}}_1 : A_1^{\text{an}} \rightarrow \mathbb{R}^{n_1}/\Lambda_1$ and $\overline{\text{val}}_2 : A_2^{\text{an}} \rightarrow \mathbb{R}^{n_2}/\Lambda_2$.

We show some assertions on abelian rank which will be needed later.

Lemma 1.4. *Let $\phi : A_1 \rightarrow A_2$ be an isogeny of abelian varieties over \mathbb{K} . Then $b(A_1) = b(A_2)$.*

Proof. It can be easily seen that there also is an isogeny $A_2 \rightarrow A_1$. In fact, for an abelian variety A , let \hat{A} denote the dual abelian variety. There is an isogeny $\alpha_i : A_i \rightarrow \hat{A}_i$ for each $i = 1, 2$. Taking the dual $\hat{\alpha}_1$ of α_1 , we have an isogeny $\hat{A}_1 \rightarrow A_1$. Therefore the composite $\alpha_2 \circ \hat{\phi} \circ \hat{\alpha}_1 : A_2 \rightarrow A_1$, where $\hat{\phi} : \hat{A}_2 \rightarrow \hat{A}_1$ is the dual isogeny of ϕ , is an isogeny.

Since we have not only an isogeny from A_1 to A_2 but also an isogeny from A_2 to A_1 , it is enough to show $b(A_1) \geq b(A_2)$, so that we show the homomorphism $\phi_{ab} : B_1 \rightarrow B_2$ associated to ϕ is surjective. Let y be a point in the abelian part B_2 of A_2 . Since $q_2^{\text{an}}|_{A_2^\circ} :$

$A_2^\circ \rightarrow B_2$, $\phi^{\text{an}} : A_1^{\text{an}} \rightarrow A_2^{\text{an}}$ and $p_1 : E_1 \rightarrow A_1^{\text{an}}$ are surjective, we can take $x \in E_1$ with $q_2^{\text{an}}(\phi^{\text{an}}(p_1(x))) = y$. Then, $\phi_{ab}(q_1^{\text{an}}(x)) = y$, which implies ϕ_{ab} is surjective. \square

Remark 1.5. Let n_i be the torus rank of A_i . If ϕ is isogeny, then $n_1 = n_2$ by the above lemma, and ϕ_{aff} is a surjection and hence an isomorphism of vector spaces. Therefore $\overline{\phi}_{\text{aff}}$ is a finite surjective homomorphism.

Proposition 1.6. *Let*

$$0 \rightarrow A_1 \rightarrow A_2 \rightarrow A_3 \rightarrow 0$$

be an exact sequence of abelian varieties over \mathbb{K} . Then we have $b(A_2) = b(A_1) + b(A_3)$, and $n(A_2) = n(A_1) + n(A_3)$, where $n(A_i)$ denote the torus rank of A_i .

Proof. We can take an abelian subvariety $A' \subset A_2$ such that the natural homomorphism $A_1 \times A' \rightarrow A_2$ given by $(a_1, a') \mapsto a_1 + a'$ is an isogeny and so is the composite $A' \hookrightarrow A_2 \rightarrow A_3$. We have $b(A_1 \times A') = b(A_1) + b(A')$ using that (1.3.12) is the Raynaud extension of the product. Taking into account Lemma 1.4, we find

$$b(A_2) = b(A_1 \times A') = b(A_1) + b(A') = b(A_1) + b(A_3)$$

as required. The other equality immediately follows from the one on the abelian ranks just obtained. \square

1.6. Mumford models, Torus-torsors, and initial degenerations. Let

$$1 \longrightarrow T \longrightarrow E \xrightarrow{g^{\text{an}}} B \longrightarrow 0$$

be the Raynaud extension of an abelian variety A over \mathbb{K} . Let n be the torus rank of A and let \mathcal{C} be a Λ -periodic Γ -rational polytopal decomposition of \mathbb{R}^n . We refer to [14, 6.1 and Appendix] for notations and conventions on convex geometry. For a polytope $\Delta \in \mathcal{C}$, the subset $\text{val}^{-1}(\Delta) \subset E$ is an analytic subdomain, and there exists a natural surjective morphism

$$q_\Delta^{\text{an}} := q^{\text{an}}|_{\text{val}^{-1}(\Delta)} : \text{val}^{-1}(\Delta) \rightarrow B.$$

(With the notation in [17, 4.7], we can write $\text{val}^{-1}(\Delta) = \bigcup_V U_{V,\Delta}$, where V runs through the formal affinoid atlas of B .) Since val induces the trivial action of T° on Δ , we have a natural T° -action on $\text{val}^{-1}(\Delta)$, which is an action over B with respect to q_Δ^{an} .

Let $\overline{\mathcal{C}}$ denote the polytopal decomposition of \mathbb{R}^n/Λ induced from \mathcal{C} by quotient, and let $\overline{\Delta} \in \overline{\mathcal{C}}$ be the polytope with a representative $\Delta \in \mathcal{C}$. Then $\overline{\text{val}}^{-1}(\overline{\Delta})$ is an analytic subdomain of A^{an} with a T° -action. The quotient map $p^{\text{an}} : E \rightarrow A^{\text{an}}$ restricts to an isomorphism $p^{\text{an}}|_{\text{val}^{-1}(\Delta)} : \text{val}^{-1}(\Delta) \rightarrow \overline{\text{val}}^{-1}(\overline{\Delta})$, via which we define

$$\overline{q}_\Delta^{\text{an}} := q_\Delta^{\text{an}} \circ (p^{\text{an}}|_{\text{val}^{-1}(\Delta)})^{-1} : \overline{\text{val}}^{-1}(\overline{\Delta}) \rightarrow B.$$

The T° -action on $\overline{\text{val}}^{-1}(\overline{\Delta})$ is a T° -action over B .

We recall some facts on *Mumford models*. See § 4, especially 4.7, in [17] for details. Gubler constructed the *Mumford model* $p : \mathcal{E} \rightarrow \mathcal{A}$ associated to \mathcal{C} . We also call \mathcal{A} the Mumford model of A . We have an extension $q : \mathcal{E} \rightarrow \mathcal{B}$ of $q^{\text{an}} : E \rightarrow B$. The T° -action on E over B extends to the \mathcal{T}° -action on \mathcal{E} over \mathcal{B} . We also have open coverings $\{\mathcal{E}_\Delta\}_{\Delta \in \mathcal{C}}$ of \mathcal{E} and $\{\mathcal{A}_\Delta\}_{\Delta \in \overline{\mathcal{C}}}$ of \mathcal{A} such that $p|_{\mathcal{E}_\Delta} : \mathcal{E}_\Delta \rightarrow \mathcal{A}_\Delta$ is an isomorphism. (With the notation in [17,

4.7], $\mathcal{E}_\Delta = \bigcup_V \mathcal{U}_{V,\Delta}$.) Note that $\overline{\text{val}}^{-1}(\overline{\Delta}) = (\mathcal{A}_\Delta)^{\text{an}}$ as analytic subspaces. The \mathcal{T}° -action on \mathcal{E} induces a \mathcal{T}° -action on \mathcal{E}_Δ over \mathcal{B} . Using the isomorphism $p|_{\mathcal{E}_\Delta}$ above, we define

$$\overline{q}_\Delta := q|_{\mathcal{E}_\Delta} \circ (p|_{\mathcal{E}_\Delta})^{-1} : \mathcal{A}_\Delta \rightarrow \mathcal{B},$$

and we have \mathcal{T}° -actions on \mathcal{E}_Δ and \mathcal{A}_Δ over \mathcal{B} . We should remark that \overline{q}_Δ depends not only on $\overline{\Delta}$ but also the choice of a representative Δ of $\overline{\Delta}$.

We also have subspaces $\text{val}^{-1}(\text{relin}(\Delta)) \subset \text{val}^{-1}(\Delta)$ and $\overline{\text{val}}^{-1}(\text{relin}(\overline{\Delta})) \subset \overline{\text{val}}^{-1}(\overline{\Delta})$ with T° -actions, where $\text{relin}(\Delta)$ and $\text{relin}(\overline{\Delta})$ denote the relative interior of Δ .⁸ We have morphisms

$$q_\Delta^{\text{an}}|_{\text{val}^{-1}(\text{relin}(\Delta))} : \text{val}^{-1}(\text{relin}(\Delta)) \rightarrow B, \quad \overline{q}_\Delta^{\text{an}}|_{\overline{\text{val}}^{-1}(\text{relin}(\overline{\Delta}))} : \overline{\text{val}}^{-1}(\text{relin}(\overline{\Delta})) \rightarrow B$$

with T° -actions over B .

Put $Z_{\text{relin}(\Delta)} := \text{red}_\mathcal{E}(\text{val}^{-1}(\text{relin}(\Delta)))$ and $Z_{\text{relin}(\overline{\Delta})} := \text{red}_\mathcal{A}(\overline{\text{val}}^{-1}(\text{relin}(\overline{\Delta})))$. Then $Z_{\text{relin}(\Delta)}$ and $Z_{\text{relin}(\overline{\Delta})}$ are the strata of $\tilde{\mathcal{E}}$ and $\tilde{\mathcal{A}}$ corresponding to $\text{relin}(\Delta)$ and $\text{relin}(\overline{\Delta})$ via the bijective correspondence described in [17, Proposition 4.8] respectively. Taking the reductions, we obtain surjective morphisms

$$(1.6.13) \quad \tilde{q}_\Delta|_{Z_{\text{relin}(\Delta)}} : Z_{\text{relin}(\Delta)} \rightarrow \tilde{\mathcal{B}}, \quad \tilde{\overline{q}}_\Delta|_{Z_{\text{relin}(\overline{\Delta})}} : Z_{\text{relin}(\overline{\Delta})} \rightarrow \tilde{\mathcal{B}}.$$

The torus $\tilde{\mathcal{T}}^\circ$ acts on $Z_{\text{relin}(\Delta)}$ and $Z_{\text{relin}(\overline{\Delta})}$, and the actions are over $\tilde{\mathcal{B}}$ with respect to \tilde{q}_Δ and $\tilde{\overline{q}}_\Delta|_{Z_{\text{relin}(\overline{\Delta})}}$ respectively. If Δ consists of one point w , then we write Z_w , $Z_{\overline{w}}$, etc. instead of $Z_{\{w\}}$ and $Z_{\{\overline{w}\}}$, etc. for simplicity. Note that $\tilde{q}_w : Z_w \rightarrow \tilde{\mathcal{B}}$ and $\tilde{\overline{q}}_w : Z_{\overline{w}} \rightarrow \tilde{\mathcal{B}}$ are $\tilde{\mathcal{T}}^\circ$ -torsors (cf. [17, Remark 4.9]).

Finally in this subsection, we define the notion of initial degenerations of a closed subvariety X of A . Let \mathcal{X}_Δ be the closure of $X^{\text{an}} \cap \overline{\text{val}}^{-1}(\overline{\Delta})$ in \mathcal{A}_Δ . We define the *initial degeneration* $\text{in}_\Delta(X)$ of X over $\overline{\Delta}$ by

$$(1.6.14) \quad \text{in}_\Delta(X) := \tilde{\mathcal{X}}_\Delta,$$

the special fiber of \mathcal{X}_Δ . It is well-defined from X and $\overline{\Delta}$. In the case that $\Delta = \{w\}$, we write $\text{in}_{\overline{w}}(X)$ for $\text{in}_\Delta(X)$. Note that $\text{in}_{\overline{w}}(X)$ is a closed subset of $Z_{\overline{w}}$.

1.7. Associated affine map. Let \mathcal{X}' be a strictly semistable formal scheme over \mathbb{K}° . We put $X' := (\mathcal{X}')^{\text{an}}$, and let $f : X' \rightarrow A^{\text{an}}$ be a morphism. We suppose that for some Mumford model \mathcal{A} of A , f extends to a morphism of formal schemes $\mathcal{X}' \rightarrow \mathcal{A}$. The following assertion is due to Gubler:

Proposition 1.7 (Proposition 5.11 in [17]). Under the setting above, there is a unique map $\overline{f}_{\text{aff}} : S(\mathcal{X}') \rightarrow \mathbb{R}^n/\Lambda$ with $\overline{f}_{\text{aff}} \circ \text{Val} = \overline{\text{val}} \circ f$ on X' . The map $\overline{f}_{\text{aff}}$ is continuous and the restriction of $\overline{f}_{\text{aff}}$ to Δ_S is an affine map for any $S \in \text{str}(\mathcal{X}')$.

We do not repeat the proof of the above proposition here, but let us recall how $\overline{f}_{\text{aff}}$ is described over Δ_S . We first fix an isomorphism between the torus part of the Raynaud extension of A and the split torus $(\mathbb{G}_m^n)^{\text{an}}$ with the standard coordinates x_1, \dots, x_n . Let the

⁸ For a polytope P , let $\text{relin}(P)$ denote the relative interior of P in this article.

same symbol $\text{val} : (\mathbb{G}_m^n)^{\text{an}} \rightarrow \mathbb{R}^n$ denote the restriction of the valuation map (1.3.7), and let $\mathbf{u} = (u_1, \dots, u_n)$ be the coordinates of \mathbb{R}^n such that $\text{val}(p) = (-\log p(x_1), \dots, -\log p(x_n))$.

We can take a open subset $\mathcal{U}' \subset \mathcal{X}'$ and an étale morphism $\psi : \mathcal{U}' \rightarrow \mathcal{S} = \mathcal{S}_1 \times \mathcal{S}_2$, where we recall $\mathcal{S}_1 = \mathbb{K}^\circ \langle x'_0, \dots, x'_r \rangle / (x'_0 \cdots x'_r - \pi)$, such that S is a distinguished stratum associated to \mathcal{U}' (cf. Proposition 1.1) and that there exists a lift $F : (\mathcal{U}')^{\text{an}} \rightarrow q^{-1}(V) \cong (\mathbb{G}_m^n)^{\text{an}} \times V$ of $f : X' \rightarrow A^{\text{an}}$, where V is a rational subdomain of B . According to [17, Proposition 2.11], we can write

$$(1.7.15) \quad F^*(x_i) = \lambda_i v_i \psi^*(x'_1)^{m_{i1}} \cdots \psi^*(x'_r)^{m_{ir}}$$

for some $\lambda_i \in \mathbb{K}^\times$, $v_i \in \mathcal{O}(\mathcal{U}')^\times$ and $\mathbf{m}_i = (m_{i1}, \dots, m_{ir}) \in \mathbb{Z}^r$. Then our \bar{f}_{aff} , via the identification (1.1.3), can be described as the composite of $f_{\text{aff}} : S(\mathcal{U}') \rightarrow \mathbb{R}^n$ with the quotient $\mathbb{R}^n \rightarrow \mathbb{R}^n / \Lambda$, where

$$(1.7.16) \quad f_{\text{aff}}(\mathbf{u}') = (\mathbf{m}_1 \cdot \mathbf{u}' + v(\lambda_1), \dots, \mathbf{m}_n \cdot \mathbf{u}' + v(\lambda_n)), \quad \mathbf{u}' \in \Delta_S = S(\mathcal{U}').$$

All m_{ij} and hence f_{aff} and \bar{f}_{aff} do not depend on the choice of a lift F .

2. TORUS-EQUIVARIANCE

The goal in this section is to show Proposition 2.3. It is a generalization of [14, Remark 6.7], in which A is totally degenerate, to general abelian varieties.

2.1. Torus-action on the formal fiber. Let \mathcal{X}' be a strictly semistable admissible formal scheme over \mathbb{K}° with the generic fiber $X' := (\mathcal{X}')^{\text{an}}$. We put $d := \dim X'$. Let $\text{red}_{\mathcal{X}'} : X' \rightarrow \tilde{\mathcal{X}}'$ be the reduction map. For a closed point $\tilde{p} \in \tilde{\mathcal{X}}'$, we put $X'_+(\tilde{p}) := (\text{red}_{\mathcal{X}'})^{-1}(\tilde{p})$ and call it the *formal fiber* over \tilde{p} . It is well known that $X'_+(\tilde{p})$ is an open subdomain of X' (cf. [14, 2.8]).

First we show that there is a natural $(\mathbb{G}_m^r)_1^{\text{an}}$ -action on $X'_+(\tilde{p})$, which is compatible with the $(\mathbb{G}_m^r)_{\mathbb{K}}$ -action mentioned in Proposition 1.3:

Lemma 2.1. *Let S be a stratum of $\tilde{\mathcal{X}}'$ of codimension r and let $\mathcal{U}' \subset \mathcal{X}'$ be an formal affine open subscheme such that S is the distinguished stratum associated to \mathcal{U}' (cf. Proposition 1.1), and fix an étale morphism $\psi : \mathcal{U}' \rightarrow \mathcal{S}$ as in (1.0.1). Then for any closed point $\tilde{p} \in S \cap \tilde{\mathcal{U}}'$, we have a $(\mathbb{G}_m^r)_1^{\text{an}}$ -action on $X'_+(\tilde{p})$ with the following properties: Let Δ_S be the canonical simplex corresponding to S . Let \mathcal{D} be a Γ -rational polytopal subdivision of the skeleton $S(\mathcal{X}')$, \mathcal{X}'' the formal model of $(\mathcal{X}')^{\text{an}}$ corresponding to \mathcal{D} , $\iota' : \mathcal{X}'' \rightarrow \mathcal{X}'$ the morphism extending the identity on X' , and let $\text{red}_{\mathcal{X}''} : X' \rightarrow \tilde{\mathcal{X}}''$ be the reduction map. Suppose that $u \in \mathcal{D}$ is a vertex with $u \in \text{relin } \Delta_S$, and let R be the stratum of $\tilde{\mathcal{X}}''$ corresponding to a vertex $u \in \mathcal{D}$. Recall that we have a $(\mathbb{G}_m^r)_{\mathbb{K}}$ -torsor $\tilde{\iota}' : R \rightarrow S$. Then we have*

$$\text{red}_{\mathcal{X}''}(X'_+(\tilde{p})) \supset \{\tilde{p}\} \times_S R,$$

and the $(\mathbb{G}_m^r)_1^{\text{an}}$ -action on $X'_+(\tilde{p})$ induces the action of $(\mathbb{G}_m^r)_{\mathbb{K}}$ on $\{\tilde{p}\} \times_S R$ mentioned in Proposition 1.3 by reduction. To be precise, the $(\mathbb{G}_m^r)_1^{\text{an}}$ -action on $X'_+(\tilde{p})$ descends to a (unique) $(\mathbb{G}_m^r)_{\mathbb{K}}$ -action on $\text{red}_{\mathcal{X}''}(X'_+(\tilde{p}))$ by the reduction maps, and this action restricts to a $(\mathbb{G}_m^r)_{\mathbb{K}}$ -action on $\{\tilde{p}\} \times_S R$, which coincides with the restriction of $(\mathbb{G}_m^r)_{\mathbb{K}}$ -action on R in Proposition 1.3.

Proof. Let us describe how this proof is organized: As a preliminary step, we will show Claim 2.1.19. After that, we will define a $(\mathbb{G}_m^r)_1^{\text{an}}$ -action on $X'_+(\tilde{p})$, and we will finally show that this action has the required properties.

We adopt the notations and conventions in § 1.3. Since our interest is local at $\tilde{p} \in S \cap \tilde{\mathcal{U}}'$, we may and do assume $\mathcal{X}' = \mathcal{U}'$. Let $p_i : \mathcal{S} = \mathcal{S}_1 \times \mathcal{S}_2 \rightarrow \mathcal{S}_i$ be the canonical projection for $i = 1, 2$. We recall that ψ and p_1 induce isomorphisms of skeletons $S(\mathcal{X}') \cong S(\mathcal{S}) \cong S(\mathcal{S}_1)$, which are all nothing but the standard simplex Δ' (cf. (1.1.2)). The Γ -rational subdivision \mathcal{D} of $S(\mathcal{X}')$ gives subdivisions on $S(\mathcal{S})$ and $S(\mathcal{S}_1)$ via these isomorphisms. Let \mathcal{S}' and \mathcal{S}'_1 be the formal scheme corresponding to these subdivisions.

We can regard $\mathcal{S}_1^{\text{an}}$ as a rational subdomain of $(\mathbb{G}_m^r)_1^{\text{an}}$ with the standard coordinates x'_1, \dots, x'_r by omitting the first coordinate x'_0 of $\mathcal{S}_1^{\text{an}}$. The affinoid torus $(\mathbb{G}_m^r)_1^{\text{an}}$ naturally acts on $\mathcal{S}_1^{\text{an}}$, and this action extends to a $(\mathbb{G}_m^r)_1^{\text{f-sch}}$ -action on \mathcal{S}_1 uniquely. Taking into account [14, 4.6], we also have a natural action of $(\mathbb{G}_m^r)_1^{\text{f-sch}}$ on \mathcal{S}'_1 . Note that $\mathcal{S}'_1 \rightarrow \mathcal{S}_1$ is $(\mathbb{G}_m^r)_1^{\text{f-sch}}$ -equivariant. On the other hand, we make $(\mathbb{G}_m^r)_1^{\text{f-sch}}$ act on \mathcal{S}_2 trivially. Then, via Cartesian diagrams

$$\begin{array}{ccc} \mathcal{S}' & \xrightarrow{p'_1} & \mathcal{S}'_1 \\ \downarrow \iota & & \downarrow \iota_1 \\ \mathcal{S} & \xrightarrow{p_1} & \mathcal{S}_1 \\ \downarrow p_2 & & \downarrow \\ \mathcal{S}_2 & \longrightarrow & \text{Spf } \mathbb{K}^\circ \end{array}$$

of formal schemes, we obtain $(\mathbb{G}_m^r)_1^{\text{f-sch}}$ -actions on \mathcal{S} and \mathcal{S}' such that the above Cartesian diagrams is $(\mathbb{G}_m^r)_1^{\text{f-sch}}$ -equivariant. Taking the reduction, we obtain a Cartesian diagram

$$(2.1.17) \quad \begin{array}{ccc} \tilde{\mathcal{S}}' & \xrightarrow{\tilde{p}'_1} & \tilde{\mathcal{S}}'_1 \\ \downarrow \tilde{\iota} & & \downarrow \tilde{\iota}_1 \\ \tilde{\mathcal{S}} & \xrightarrow{\tilde{p}_1} & \tilde{\mathcal{S}}_1 \end{array}$$

of $(\mathbb{G}_m^r)_{\tilde{\mathbb{K}}}$ -schemes.

Let $\text{val}' : (\mathbb{G}_m^r)^{\text{an}} \rightarrow \mathbb{R}^r$ be the valuation map. We know that val' induces $S(\mathcal{S}_1) \cong \Delta'$, as recalled in § 1.3. We put $u_1 := p_1^{\text{an}}(\psi^{\text{an}}(u)) \in S(\mathcal{S}_1)$. It is a vertex in \mathcal{D}_1 , where \mathcal{D}_1 denote the subdivision of $S(\mathcal{S}'_1)$ associated to \mathcal{D} via the isomorphism $S(\mathcal{X}') \cong S(\mathcal{S}_1)$. Let T'_1 be the stratum of $\tilde{\mathcal{S}}'_1$ corresponding to u_1 . Note that $\tilde{\iota}_1(T'_1) = \{\tilde{o}\}$ since u_1 sits in the interior of the maximal simplex of $S(\mathcal{S}_1)$ (cf. Proposition 1.3). Recall that T'_1 is given by $T'_1 = \text{red}_{\mathcal{S}'_1}((\text{val}'|_{\mathcal{S}_1^{\text{an}}})^{-1}(\text{val}'(u_1)))$. Since val' is invariant by the $(\mathbb{G}_m^r)_1^{\text{an}}$ -action on $\mathcal{S}_1^{\text{an}}$, we find that $(\text{val}')^{-1}(\text{val}'(u_1))$ is stable by the action of $(\mathbb{G}_m^r)_1^{\text{an}}$, and hence that T'_1 is stable under the action of $(\mathbb{G}_m^r)_{\tilde{\mathbb{K}}}$ on $\tilde{\mathcal{S}}'_1$. Restricting $\tilde{\iota}_1 : \tilde{\mathcal{S}}'_1 \rightarrow \tilde{\mathcal{S}}_1$ in the square (2.1.17) to $T'_1 \rightarrow \{\tilde{o}\}$,

we obtain a Cartesian diagram

$$\begin{array}{ccc} \left(\tilde{p}'_1\right)^{-1}(T'_1) & \longrightarrow & T'_1 \\ \tilde{\iota}|_{\left(\tilde{p}'_1\right)^{-1}(T'_1)} \downarrow & & \downarrow \tilde{\iota}_1|_{T'_1} \\ (\tilde{p}_1)^{-1}(\tilde{o}) & \longrightarrow & \{\tilde{o}\} \end{array}$$

of $(\mathbb{G}_m^r)_{\mathbb{K}}$ -varieties. Note that the $(\mathbb{G}_m^r)_{\mathbb{K}}$ -action on $(\tilde{p}_1)^{-1}(\tilde{o})$ is trivial since the $(\mathbb{G}_m^r)_{\mathbb{K}}$ -action on $\tilde{\mathcal{S}}_2$ is trivial.

As in [17, Remark 5.6], we have a Cartesian diagram

$$(2.1.18) \quad \begin{array}{ccc} \mathcal{X}'' & \xrightarrow{\psi'} & \mathcal{S}' \\ \iota' \downarrow & & \downarrow \iota \\ \mathcal{X}' & \xrightarrow{\psi} & \mathcal{S}. \end{array}$$

As we can find in the proof of [17, Proposition 5.7], we have $S = (\tilde{p}_1 \circ \tilde{\psi})^{-1}(\tilde{o})$ and $R = \left(\tilde{p}'_1 \circ \tilde{\psi}'\right)^{-1}(T'_1)$. Therefore we obtain a Cartesian diagram

$$\begin{array}{ccc} R & \xrightarrow{\tilde{\psi}'|_R} & \left(\tilde{p}'_1\right)^{-1}(T'_1) \\ \tilde{\iota}'|_R \downarrow & & \downarrow \tilde{\iota}|_{\left(\tilde{p}'_1\right)^{-1}(T'_1)} \\ S & \xrightarrow{\tilde{\psi}|_S} & (\tilde{p}_1)^{-1}(\tilde{o}). \end{array}$$

We make $(\mathbb{G}_m^r)_{\mathbb{K}}$ act on S trivially. Then the bottom row in the above diagram is $(\mathbb{G}_m^r)_{\mathbb{K}}$ -equivariant, and hence we can put an $(\mathbb{G}_m^r)_{\mathbb{K}}$ -action on R by the Cartesian product. That makes $R \rightarrow S$ a $(\mathbb{G}_m^r)_{\mathbb{K}}$ -torsor, and this $(\mathbb{G}_m^r)_{\mathbb{K}}$ -action is nothing but the torsor structure in [17, Corollary 5.9] and hence that mentioned in Proposition 1.3. In particular, we have the following:

Claim 2.1.19. We have a $(\mathbb{G}_m^r)_{\mathbb{K}}$ -equivariant isomorphism

$$\{\tilde{p}\} \times_S R \cong \left(\tilde{\iota}|_{\left(\tilde{p}'_1\right)^{-1}(T'_1)}\right)^{-1}(\tilde{\psi}(\tilde{p})),$$

where the $(\mathbb{G}_m^r)_{\mathbb{K}}$ -action on the left-hand side is the restriction of the $(\mathbb{G}_m^r)_{\mathbb{K}}$ -action in Proposition 1.3, and that of the right-hand side is given by the reduction of $(\mathbb{G}_m^r)_1^{\text{an}}$ -action on \mathcal{S}^{an} .

Now we define a $(\mathbb{G}_m^r)_1^{\text{an}}$ action on $X'_+(\tilde{p})$. We put $(\mathcal{S}^{\text{an}})_+(\tilde{\psi}(\tilde{p})) := \text{red}_{\mathcal{S}}^{-1}(\tilde{\psi}(\tilde{p}))$. Since $\tilde{\psi}(\tilde{p})$ is fixed by the action of $(\mathbb{G}_m^r)_{\mathbb{K}}$, we find that $(\mathbb{G}_m^r)_1^{\text{an}}$ acts on $(\mathcal{S}^{\text{an}})_+(\tilde{\psi}(\tilde{p}))$ and that $(\mathbb{G}_m^r)_{\mathbb{K}}$ acts on $\text{red}_{\mathcal{S}'}((\mathcal{S}^{\text{an}})_+(\tilde{\psi}(\tilde{p}))) \subset \tilde{\mathcal{S}}'_1$ by reduction. By virtue of [4, Lemma 4.4] or [17, Proposition 2.9], the restricted morphism

$$(2.1.20) \quad \psi'^{\text{an}} : X'_+(\tilde{p}) \rightarrow (\mathcal{S}^{\text{an}})_+(\tilde{\psi}(\tilde{p}))$$

is an isomorphism. Through this isomorphism, we define a $(\mathbb{G}_m^r)_1^{\text{an}}$ -action on $X'_+(\tilde{p})$.

To complete the proof, we show that this action satisfies the required conditions.

Claim 2.1.21. The $(\mathbb{G}_m^r)_1^{\text{an}}$ -action on $X'_+(\tilde{p})$ induces an $(\mathbb{G}_m^r)_{\mathbb{K}}$ -action on $\text{red}_{\mathcal{X}''}(X'_+(\tilde{p}))$ by reduction, and the morphism

$$(2.1.22) \quad \tilde{\psi}' : \text{red}_{\mathcal{X}''}(X'_+(\tilde{p})) \rightarrow \text{red}_{\mathcal{S}'}((\mathcal{S}^{\text{an}})_+(\tilde{\psi}(\tilde{p})))$$

is a $(\mathbb{G}_m^r)_{\mathbb{K}}$ -equivariant isomorphism.

Proof. Let us first show that (2.1.22) is an isomorphism. Note that we have an isomorphism $\tilde{\psi}' : (\tilde{\iota}')^{-1}(\tilde{p}) \cong \tilde{\iota}^{-1}(\tilde{\psi}(\tilde{p}))$ coming from the Cartesian diagram (2.1.18). It follows therefore from the diagram

$$\begin{array}{ccc} \text{red}_{\mathcal{X}''}(X'_+(\tilde{p})) & \xrightarrow{\tilde{\psi}'} & \text{red}_{\mathcal{S}'}((\mathcal{S}^{\text{an}})_+(\tilde{\psi}(\tilde{p}))) \\ \downarrow \cap & & \downarrow \cap \\ (\tilde{\iota}')^{-1}(\tilde{p}) & \xrightarrow{\tilde{\psi}'} & \tilde{\iota}^{-1}(\tilde{\psi}(\tilde{p})) \end{array}$$

that the morphism (2.1.22) is injective. On the other hand, the isomorphism $X'_+(\tilde{p}) \cong (\mathcal{S}^{\text{an}})_+(\tilde{\psi}(\tilde{p}))$ tells us that (2.1.22) is surjective because the reduction maps are surjective. Thus we see that it is an isomorphism.

Recall that our $(\mathbb{G}_m^r)_{\mathbb{K}}$ -action on $\text{red}_{\mathcal{S}'}((\mathcal{S}^{\text{an}})_+(\tilde{\psi}(\tilde{p})))$ is given by the reduction of the $(\mathbb{G}_m^r)_1^{\text{an}}$ -action on $(\mathcal{S}^{\text{an}})_+(\tilde{\psi}(\tilde{p}))$, and that the $(\mathbb{G}_m^r)_1^{\text{an}}$ -action on $X'_+(\tilde{p})$ is given via the isomorphism (2.1.20). Then it follows from a commutative diagram

$$\begin{array}{ccc} X'_+(\tilde{p}) & \xrightarrow{\cong} & (\mathcal{S}^{\text{an}})_+(\tilde{\psi}(\tilde{p})) \\ \text{red}_{\mathcal{X}''} \downarrow & & \downarrow \text{red}_{\mathcal{S}'} \\ \text{red}_{\mathcal{X}''}(X'_+(\tilde{p})) & \xrightarrow{\cong} & \text{red}_{\mathcal{S}'}((\mathcal{S}^{\text{an}})_+(\tilde{\psi}(\tilde{p}))) \end{array}$$

that the $(\mathbb{G}_m^r)_{\mathbb{K}}$ -action on $\text{red}_{\mathcal{X}''}(X'_+(\tilde{p}))$ induced via (2.1.22) coincides with the one induced from the $(\mathbb{G}_m^r)_1^{\text{an}}$ -action on $X'_+(\tilde{p})$ by reduction. Thus we obtain our claim. \square

It is obvious from the definition that $\{\tilde{p}\} \times_S R \hookrightarrow \text{red}_{\mathcal{X}''}(X'_+(\tilde{p}))$. What is remaining to be shown is that this inclusion map is $(\mathbb{G}_m^r)_{\mathbb{K}}$ -equivariant with respect to the $(\mathbb{G}_m^r)_{\mathbb{K}}$ -action on $\{\tilde{p}\} \times_S R$ in Proposition 1.3 and $(\mathbb{G}_m^r)_{\mathbb{K}}$ -action on $\text{red}_{\mathcal{X}''}(X'_+(\tilde{p}))$ obtained in Claim 2.1.21, but it is not difficult. In fact, let us look at the following commutative diagram:

$$\begin{array}{ccc} \text{red}_{\mathcal{X}''}(X'_+(\tilde{p})) & \xrightarrow[\cong]{\tilde{\psi}'} & \text{red}_{\mathcal{S}'}((\mathcal{S}^{\text{an}})_+(\tilde{\psi}(\tilde{p}))) \\ \uparrow \cup & & \uparrow \cup \\ \{\tilde{p}\} \times_S R & \xrightarrow{\cong} & \left(\tilde{\iota}|_{(\tilde{p}_1')^{-1}(T_1')} \right)^{-1}(\tilde{\psi}(\tilde{p})). \end{array}$$

Claim 2.1.21 tells us that the first row is $(\mathbb{G}_m^r)_{\mathbb{K}}$ -equivariant. The second row is $(\mathbb{G}_m^r)_{\mathbb{K}}$ -equivariant by Claim 2.1.19. The inclusion in the right column is $(\mathbb{G}_m^r)_{\mathbb{K}}$ -equivariant because their actions are both given by the reduction of the restriction of the $(\mathbb{G}_m^r)_1^{\text{an}}$ -action on \mathcal{S}^{an} . Consequently, we conclude that the inclusion $\{\tilde{p}\} \times_S R \rightarrow \text{red}_{\mathcal{X}''}(X'_+(\tilde{p}))$ in the left-hand side is also $(\mathbb{G}_m^r)_{\mathbb{K}}$ -equivariant. That shows that there is a $(\mathbb{G}_m^r)_1^{\text{an}}$ -action on $X'_+(\tilde{p})$ which

induces by reduction a $(\mathbb{G}_m^r)_{\mathbb{K}}^{\text{an}}$ -action on $\{\tilde{p}\} \times_S R$ compatible with the one mentioned in Proposition 1.3. Thus we obtain our lemma. \square

2.2. Torus-equivariance between strata. Let A be an abelian variety over \mathbb{K} of torus rank n and we fix an identification between the torus part of A and $(\mathbb{G}_m^n)^{\text{an}}$. We recall here that Λ is a lattice of \mathbb{R}^n associated to the kernel of the Raynaud extension $E \rightarrow A^{\text{an}}$ of A , and that $\overline{\text{val}} : A^{\text{an}} \rightarrow \mathbb{R}^n/\Lambda$ is the valuation map. From now on, we fix a Λ -periodic Γ -rational polytopal decomposition \mathcal{C}_0 of \mathbb{R}^n . Let $\overline{\mathcal{C}}_0$ denote the polytopal decomposition of \mathbb{R}^n/Λ induce from \mathcal{C}_0 by quotient. Let $p_0 : \mathcal{E}_0 \rightarrow \mathcal{A}_0$ be the Mumford model associated to \mathcal{C}_0 . Let \mathcal{X}' be a connected strictly semistable formal scheme over \mathbb{K}° and let $\varphi_0 : \mathcal{X}' \rightarrow \mathcal{A}_0$ be a morphism. We put $X' := (\mathcal{X}')^{\text{an}}$ with $d := \dim X'$, and let $f : X' \rightarrow A^{\text{an}}$ be the generic fiber of φ_0 .

Let S be a stratum of $\tilde{\mathcal{X}}'$. We choose a formal open subscheme $\mathcal{U}' \subset \mathcal{X}'$ such that S is a distinguished stratum associated to \mathcal{U}' as in Proposition 1.1, and fix an étale morphism $\psi : \mathcal{U}' \rightarrow \mathcal{S}$ as in (1.0.1). Let m_{ij} , for $1 \leq i \leq n$ and $1 \leq j \leq r$, be integers which give the linear map f_{aff} (cf. (1.7.16)). We define a homomorphism $h_{f, \Delta_S} : (\mathbb{G}_m^r)_1^{\text{an}} \rightarrow (\mathbb{G}_m^n)_1^{\text{an}}$ by

$$h_{f, \Delta_S}^*(x_i) = (x'_1)^{m_{i1}} \cdots (x'_r)^{m_{ir}}, \quad i = 1, \dots, n.$$

Since $(\mathbb{G}_m^n)_1^{\text{an}}$ acts on A^{an} naturally, we can make $(\mathbb{G}_m^r)_1^{\text{an}}$ act on A^{an} via h_{f, Δ_S} .

Lemma 2.2. *With the notation above, let $\tilde{p} \in S \cap \tilde{\mathcal{U}}'$ be a closed point. Recall that we put $X'_+(\tilde{p}) := \text{red}_{\tilde{\mathcal{X}'}}^{-1}(\tilde{p})$. Then the morphism*

$$f|_{X'_+(\tilde{p})} : X'_+(\tilde{p}) \rightarrow A^{\text{an}}$$

is $(\mathbb{G}_m^r)_1^{\text{an}}$ -equivariant with respect to the $(\mathbb{G}_m^r)_1^{\text{an}}$ -action on $X'_+(\tilde{p})$ given in Lemma 2.1 and that on A^{an} induced by h_{f, Δ_S} .

Proof. First we note that $X'_+(\tilde{p}) \subset (\mathcal{U}')^{\text{an}}$ since $\tilde{p} \in \tilde{\mathcal{U}}'$. As we saw in § 1.7, we have a commutative diagram

$$\begin{array}{ccc} (\mathcal{U}')^{\text{an}} & \xrightarrow{F} & (\mathbb{G}_m^n)_1^{\text{an}} \times V \\ \uparrow \cup & & \cap \downarrow \\ X'_+(\tilde{p}) & \xrightarrow{f} & A^{\text{an}}, \end{array}$$

where we adopt the notation in § 1.7. That reduces us to show that F is a $(\mathbb{G}_m^r)_1^{\text{an}}$ -equivariant morphism. The $(\mathbb{G}_m^r)_1^{\text{an}}$ -action on $X'_+(\tilde{p})$ in Lemma 2.1 is defined thought the isomorphism (2.1.20). Then the $(\mathbb{G}_m^r)_1^{\text{an}}$ -equivariance of F follows straightforwardly from the description of (1.7.15) and the definition of h_{f, Δ_S} . \square

Let \mathcal{C} be a Λ -periodic Γ -rational polytopal subdivision of \mathcal{C}_0 , with the induced polytopal subdivision $\overline{\mathcal{C}}$ of $\overline{\mathcal{C}}_0$, and let $p : \mathcal{E} \rightarrow \mathcal{A}$ be the Mumford model of A associated to \mathcal{C} . Let \mathcal{D} be the subdivision of $S(\mathcal{X}')$ given by

$$\mathcal{D} := \left\{ \Delta_S \cap \overline{f}_{\text{aff}}^{-1}(\overline{\Delta}) \mid S \in \text{str}(\tilde{\mathcal{X}}'), \overline{\Delta} \in \overline{\mathcal{C}} \right\}$$

and let \mathcal{X}'' be the formal scheme associated to the formal analytic structure corresponding to \mathcal{D} . Then we have a morphism $\varphi' : \mathcal{X}'' \rightarrow \mathcal{A}$ extending $f : X' \rightarrow A^{\text{an}}$ by [17, Proposition 5.14].

Proposition 2.3. *Let $u \in \mathcal{D}$ be a vertex, Δ_S the canonical simplex of $S(\mathcal{X}')$ with $u \in \text{relin} \Delta_S$ and let R be the stratum of \mathcal{X}'' corresponding to u (cf. Proposition 1.2). Let $\overline{\Delta} \in \overline{\mathcal{C}}$ be the polytope with $\overline{f}_{\text{aff}}(u) \in \text{relin}(\overline{\Delta})$. Take a representative $\Delta \in \mathcal{C}$ of $\overline{\Delta}$, and let $\tilde{q}_{\overline{\Delta}}|_{Z_{\text{relin} \overline{\Delta}}}$ be the morphism in (1.6.13). Then, there exists a unique morphism $\beta_{\Delta} : S \rightarrow \tilde{\mathcal{B}}$ such that the diagram*

$$\begin{array}{ccc} R & \xrightarrow{\tilde{\varphi}'} & Z_{\text{relin} \overline{\Delta}} \\ \tilde{\iota}' \downarrow & & \downarrow \tilde{q}_{\overline{\Delta}}|_{Z_{\text{relin} \overline{\Delta}}} \\ S & \xrightarrow{\beta_{\Delta}} & \tilde{\mathcal{B}} \end{array}$$

commutes. Moreover, this diagram is $(\mathbb{G}_m^r)_{\mathbb{K}}$ -equivariant, with respect to the $(\mathbb{G}_m^r)_{\mathbb{K}}$ -action on R in Proposition 1.3, that on $Z_{\text{relin} \overline{\Delta}}$ induced by \tilde{h}_{f, Δ_S} and the trivial actions on S and $\tilde{\mathcal{B}}$.

Proof. Let us define the morphism $S \rightarrow \tilde{\mathcal{B}}$ first. Since there is a local section of $\tilde{\iota}'$, we can define locally on S a morphism from S to $\tilde{\mathcal{B}}$ which is compatible with $\tilde{q}_{\overline{\Delta}}|_{Z_{\text{relin} \overline{\Delta}}} \circ \tilde{\varphi}'$. Since the fiber of $\tilde{\iota}'$ is an algebraic torus and $\tilde{\mathcal{B}}$ is an abelian variety, any fiber contracts to a point by $\tilde{q}_{\overline{\Delta}}|_{Z_{\text{relin} \overline{\Delta}}} \circ \tilde{\varphi}'$. That implies that the local morphism from S to $\tilde{\mathcal{B}}$ defined just above does not depend on the choice of local sections of $\tilde{\iota}'$. Accordingly, the local morphisms patch together to be a global morphism $S \rightarrow \tilde{\mathcal{B}}$, which satisfies the commutativity of the diagram. The uniqueness is clear from the construction.

It now remains only to show that $\tilde{\varphi}' : R \rightarrow Z_{\text{relin} \overline{\Delta}}$ is $(\mathbb{G}_m^r)_{\mathbb{K}}$ -equivariant. Take an arbitrary $\tilde{p} \in S \cap \tilde{\mathcal{U}}'$. We have relations $R = \text{red}_{\mathcal{X}''}(\text{Val}^{-1}(u))$ and $\{u\} = \text{Val}((\text{red}_{\mathcal{X}''})^{-1}(\{\tilde{p}\} \times_S R))$ by Proposition 1.2. Since

$$\overline{\text{val}}(f((\text{red}_{\mathcal{X}''})^{-1}(\{\tilde{p}\} \times_S R))) = \overline{f}_{\text{aff}}(\text{Val}((\text{red}_{\mathcal{X}''})^{-1}(\{\tilde{p}\} \times_S R))) = \overline{f}_{\text{aff}}(u) \in \text{relin} \overline{\Delta},$$

we see that f can be restricted to a morphism

$$(2.3.23) \quad (\text{red}_{\mathcal{X}''})^{-1}(\{\tilde{p}\} \times_S R) \rightarrow \overline{\text{val}}^{-1}(\text{relin}(\overline{\Delta})).$$

Since $\{\tilde{p}\} \times_S R$ is stable under the $(\mathbb{G}_m^r)_{\mathbb{K}}$ -action, the subspace $(\text{red}_{\mathcal{X}''})^{-1}(\{\tilde{p}\} \times_S R)$ of $X'_+(\tilde{p})$ is stable under the $(\mathbb{G}_m^r)_1^{\text{an}}$ -action. Accordingly $(\mathbb{G}_m^r)_1^{\text{an}}$ acts on $(\text{red}_{\mathcal{X}''})^{-1}(\{\tilde{p}\} \times_S R)$, and by virtue of Lemma 2.2, we find that the morphism (2.3.23) is $(\mathbb{G}_m^r)_1^{\text{an}}$ -equivariant. Taking the reduction with respect to $\text{red}_{\mathcal{X}''}$ and $\text{red}_{\mathcal{A}}$, we find that $\{\tilde{p}\} \times_S R \rightarrow Z_{\text{relin} \overline{\Delta}}$ is $(\mathbb{G}_m^r)_{\mathbb{K}}$ -equivariant with respect to our actions by virtue of Lemma 2.1. Since $S \cap \tilde{\mathcal{U}}'$ is dense in S and $\tilde{p} \in S \cap \tilde{\mathcal{U}}'$ is arbitrarily chosen, that means that $R \rightarrow Z_{\text{relin} \overline{\Delta}}$ is $(\mathbb{G}_m^r)_{\mathbb{K}}$ -equivariant. Thus we prove our assertion. \square

3. INITIAL DEGENERATION AND NON-DEGENERATE STRATA

We adopt the same notations and convention as those fixed in § 2.2.

3.1. Torus parallel to a polytope. Let $\overline{\sigma} \subset \mathbb{R}^n/\Lambda$ be a Γ -rational polytope. We recall the notion of the *torus parallel* to $\overline{\sigma}$. First we note that our $(\mathbb{G}_m^r)_{\mathbb{K}}$ is identified with $\text{Spec}(\mathbb{K}[M])$, where $N = \mathbb{Z}^n$ and $M := N^*$. Let $\sigma \subset \mathbb{R}^n = N_{\mathbb{R}}$ be a polytope which is a lift of $\overline{\sigma}$, and let $\mathbb{L}_{\overline{\sigma}}$ be the linear subspace spanned by $\sigma - \mathbf{u}$ for $\mathbf{u} \in \sigma$. $\mathbb{L}_{\overline{\sigma}}$ depends only on $\overline{\sigma}$. Set

$M'_{\bar{\sigma}} := (\mathbb{L}_{\bar{\sigma}} \cap N)^\perp \subset M$. Then $\text{Spec}(\mathbb{K}[M'_{\bar{\sigma}}])$ is an algebraic torus of dimension $n - \dim \bar{\sigma}$ and we have a canonical surjective homomorphism $\text{Spec}(\mathbb{K}[M]) \rightarrow \text{Spec}(\mathbb{K}[M'_{\bar{\sigma}}])$. We define the *torus parallel* to $\bar{\sigma}$, to be its kernel, which is denoted by $\mathbb{T}''_{\bar{\sigma}}$. Note that $\dim \mathbb{T}''_{\bar{\sigma}} = \dim \sigma$.

Let S be a strata of $\tilde{\mathcal{X}}'$ of codimension r . Again we take be an formal affine open subscheme $\mathcal{U}' \subset \mathcal{X}'$ such that S is the distinguished stratum associated to $\tilde{\mathcal{U}}'$ (cf. Proposition 1.1), and fix an étale morphism $\psi : \mathcal{U}' \rightarrow \mathcal{S}$ as in (1.0.1). Let Δ_S be the canonical simplex in the skeleton $S(\mathcal{X}')$ corresponding to S . We consider $\mathbb{T}''_{\bar{\sigma}}$ in the case of $\bar{\sigma} := \bar{f}_{\text{aff}}(\Delta_S)$. We express \bar{f}_{aff} as in (1.7.16), putting

$$L(\bar{f}_{\text{aff}}) := \begin{pmatrix} m_{11} & \cdots & m_{1r} \\ \vdots & \cdots & \vdots \\ m_{n1} & \cdots & m_{nr} \end{pmatrix},$$

the matrix corresponding to f_{aff} . We can describe the torus parallel to $\bar{\sigma}$ in terms of $L(\bar{f}_{\text{aff}})$ like this: The linear subspace $\mathbb{L}_{\bar{\sigma}}$ is equal to the image of f_{aff} . Then we can see that $\mathbb{T}''_{\bar{\sigma}}$ coincides with the image of the homomorphism

$$(t_1, \dots, t_r) \mapsto (t_1^{m_{11}} \cdots t_r^{m_{1r}}, \dots, t_1^{m_{n1}} \cdots t_r^{m_{nr}}),$$

namely, $\mathbb{T}''_{\bar{\sigma}} = \text{Image } h_{f, \Delta_S}$.

3.2. Initial degenerations and non-degenerate strata. Gubler defined the notion of *non-degenerate strata* with respect to f for canonical simplices in [17, 6,3], which we recall here. The morphism $\varphi_0 : \mathcal{X}' \rightarrow \mathcal{A}_0$ in § 2.2 gives us a morphism $\tilde{\varphi}_0 : S \rightarrow \tilde{\mathcal{A}}_0$ by taking the special fibers. Then [17, Lemma 5.15] tells us that there is a lift $\tilde{\Phi}_0 : S \rightarrow \tilde{\mathcal{E}}_0$ of $\tilde{\varphi}_0$. We say a canonical simplex Δ_S is *non-degenerate with respect to f* if $\dim \bar{f}_{\text{aff}}(\Delta_S) = \dim \Delta_S$ and $\dim \tilde{\Phi}_0(\tilde{q}_0(S)) = \dim S$, where $q_0 : \mathcal{E}_0 \rightarrow \mathcal{B}$ is the surjective morphism in the Mumford model of the Raynaud extension (cf. § 1.6). It does not depend on the choice of $\tilde{\Phi}_0$.

We would like to describe the condition of $\dim \tilde{\Phi}_0(\tilde{q}_0(S)) = \dim S$ by using β_Δ , when we are in the situation of Proposition 2.3. We consider a morphism $\iota_0 : \mathcal{E} \rightarrow \mathcal{E}_0$ and $\bar{\iota}_0 : \mathcal{A} \rightarrow \mathcal{A}_0$ extending the identities, with the notation in § 2.2. Let $\Delta \in \mathcal{C}$ be a polytope taken in Proposition 2.3 and let $\Delta_0 \in \mathcal{C}_0$ be the polytope with $\text{relin } \Delta \subset \text{relin } \Delta_0$. Then $\tilde{\iota}_0$ restricts to $Z_{\text{relin } \bar{\Delta}} \rightarrow Z_{\bar{\Delta}_0}$, and we have a commutative diagram

$$\begin{array}{ccccccc} R & \xrightarrow{\tilde{\varphi}'} & Z_{\text{relin } \bar{\Delta}} & \xleftarrow[\cong]{\tilde{p}|_{Z_{\text{relin } \Delta}}} & Z_{\text{relin } \Delta} & \xrightarrow{\tilde{q}} & \tilde{\mathcal{B}} \\ \downarrow \tilde{\iota}' & & \downarrow \tilde{\iota}_0 & & \downarrow \tilde{\iota}_0 & & \downarrow \text{id} \\ S & \xrightarrow{\tilde{\varphi}_0} & Z_{\text{relin } \bar{\Delta}_0} & \xleftarrow[\cong]{\tilde{p}_0|_{Z_{\text{relin } \Delta_0}}} & Z_{\text{relin } \Delta_0} & \xrightarrow{\tilde{q}_0} & \tilde{\mathcal{B}}, \end{array}$$

where $q : \mathcal{E} \rightarrow \mathcal{B}$ is the morphism as in § 1.6. We can take a lift $\tilde{\Phi}_0$ of $\tilde{\varphi}_0$ such that $\tilde{\Phi}_0(S) \subset Z_{\text{relin } \Delta_0}$, and we see that

$$\tilde{q}_0 \circ \tilde{\Phi}_0 \circ \tilde{\iota}' = \tilde{q} \circ (\tilde{p}|_{Z_{\text{relin } \Delta}})^{-1} \circ \tilde{\varphi}' = \tilde{q}_{\bar{\Delta}} \circ \tilde{\varphi}' = \beta_\Delta \circ \tilde{\iota}',$$

which concludes that $\beta_\Delta = \tilde{q}_0 \circ \tilde{\Phi}_0$. Accordingly we find that, in the setting of Proposition 2.3, Δ_S is non-degenerate with respect to f if and only if $\dim \bar{f}_{\text{aff}}(\Delta_S) = \dim \Delta_S$ and $\dim \beta_\Delta(S) = \dim S$.

Remark 3.1. Let \mathcal{X}'' be the formal model of X' corresponding to \mathcal{D} and let u be a vertex of a subdivision \mathcal{D} of $S(\mathcal{X}')$, with the corresponding stratum R of \mathcal{X}'' . Let us consider the diagram in Proposition 2.3 in the case of $\dim \overline{\Delta} = 0$, that is, $\overline{\Delta} = \{\overline{w}\} = \{\overline{f}_{\text{aff}}(u)\}$. Then, we can see $\dim \tilde{\varphi}'(R) = d$ if Δ_S is non-degenerate with respect to f . In fact, the torus \mathbb{T}'' parallel to $\overline{f}_{\text{aff}}(\Delta_S)$ is equal to the image of \tilde{h}_{f, Δ_S} , and hence it is of dimension $\dim \overline{f}_{\text{aff}}(\Delta_S)$. Moreover, we find that \mathbb{T}'' acts on $\tilde{\varphi}'(R)$ by Proposition 2.3, and this action is free since the $(\mathbb{G}_m^n)_{\mathbb{K}}$ -action on $Z_{\overline{w}}$ is free as remarked in § 1.6. Since any \mathbb{T}'' -orbit contracts to a point by $\tilde{q}_w : Z_{\overline{w}} \rightarrow \tilde{\mathcal{B}}$ in (1.6.13), where $w \in \mathbb{R}^n$ is a representative of $\overline{w} \in \mathbb{R}^n/\Lambda$, we have $\dim \tilde{\varphi}'(R) \geq \dim \tilde{q}_w(\tilde{\varphi}'(R)) + \dim \overline{f}_{\text{aff}}(\Delta_S)$. By our assumption of non-degeneracy of Δ_S with respect to f , we have $\dim \overline{f}_{\text{aff}}(\Delta_S) = \dim \Delta_S$, and $\dim S = \dim \beta_w(S)$, the latter of which equals $\dim \tilde{q}_w(\tilde{\varphi}'(R))$. Therefore, we have

$$d \geq \dim \tilde{\varphi}'(R) \geq \dim \tilde{q}_w(\tilde{\varphi}'(R)) + \dim \overline{f}_{\text{aff}}(\Delta_S) = \dim S + \dim \Delta_S = \dim R = d$$

and hence we obtain $\dim \tilde{\varphi}'(R) = d$.

Lemma 3.2. *Let u be a vertex of a subdivision \mathcal{D} of $S(\mathcal{X}')$ and let R be the stratum of \mathcal{X}'' corresponding to u . Let $\xi_R \in (\mathcal{X}'')^{\text{an}}$ be the point corresponding to the generic point of R . Then $u = \xi_R$.*

Proof. It follows from [17, Proposition 5.7 and Corollary 5.9 (a)] that $u = \text{Val}(\xi_R)$. Since Val restricts to the identity on $S(\mathcal{X}')$, it remains to show $\xi_R \in S(\mathcal{X}'')$, but it is done in the proof of [17, Corollary 5.9 (g)]. \square

Let X be an irreducible closed subvariety of A . Recall that $\text{in}_{\overline{w}}(X)$ denotes the initial degeneration over a Γ -rational point $\overline{w} \in \mathbb{R}^n/\Lambda$ (cf. (1.6.14)). Let W be an irreducible component of $\text{in}_{\overline{w}}(X)$. Since $W \subset \text{in}_{\overline{w}}(X) \subset Z_{\overline{w}}$ and we have a map $\tilde{q}_w : Z_{\overline{w}} \rightarrow \tilde{\mathcal{B}}$ for a $w \in \mathbb{R}^n$ over \overline{w} , we can consider the image $\tilde{q}_w(W) \subset \tilde{\mathcal{B}}$. We put $b(W) := \dim \tilde{q}_w(W)$ and call it the *dimension of the abelian part* of W .

Now we can state one of the most important assertions, which will play a crucial role in the proof of Theorem 4.5.

Proposition 3.3. *Let X be a d -dimensional closed subvariety of an abelian variety A and let \mathcal{X}_0 be the closure of X in \mathcal{A}_0 . Suppose that our morphism $\varphi_0 : \mathcal{X}' \rightarrow \mathcal{A}_0$ factors through \mathcal{X}_0 to be a semistable alteration for \mathcal{X}_0 .⁹ Let $f : X' \rightarrow A^{\text{an}}$ be the generic fiber of φ_0 . For a Γ -rational point $\overline{w} \in \overline{\text{val}}(X^{\text{an}})$, let W be an irreducible component of $\text{in}_{\overline{w}}(X)$, and let $\xi_W \in X^{\text{an}}$ be a point corresponding to the generic point of $W \subset \mathcal{X}_0$. Let u be a point in $S(\mathcal{X}')$ and let S be a stratum of \mathcal{X}' such that $f(u) = \xi_W$ and $u \in \text{relin}(\Delta_S)$. Let \mathbb{T}'' be the torus parallel to $\overline{f}_{\text{aff}}(\Delta_S)$. Then, \mathbb{T}'' acts freely on W , and furthermore, the following statements are equivalent to each other, in which $\Xi := W/\mathbb{T}''$.*

- (a) Δ_S is non-degenerate with respect to f .
- (b) $\dim \Xi = b(W)$.
- (c) $\dim \Xi \leq b(W)$.

⁹One can find the definition of semistable alteration in § 4.1.

Proof. Let $\overline{\mathcal{C}}$ be a Γ -rational subdivision of $\overline{\mathcal{C}}_0$ such that \overline{w} itself is a vertex of $\overline{\mathcal{C}}$. Let \mathcal{X}'' be the formal model of X' associated to the polytopal subdivision

$$\mathcal{D} := \left\{ \Delta_{S'} \cap \overline{f}_{\text{aff}}^{-1}(\overline{\Delta}) \mid S' \in \text{str}(\tilde{\mathcal{X}}'), \overline{\Delta} \in \overline{\mathcal{C}} \right\}$$

of $S(\mathcal{X}')$, and let $\iota' : \mathcal{X}'' \rightarrow \mathcal{X}'$ denote the natural morphism extending the identity on the generic fiber. Let $R \in \text{str}(\tilde{\mathcal{X}}'')$ be the stratum corresponding to u and put $r := \dim \Delta_S$. We know $\tilde{\iota}'(R) = S$. We fix a formal affine open subscheme $\mathcal{U}' \subset \mathcal{X}'$ with an étale morphism as in (1.0.1) such that S is a distinguished stratum associated to \mathcal{U}' . Then $(\mathbb{G}_m^r)_{\mathbb{K}}$ acts on R , which makes $\tilde{\iota}' : R \rightarrow S$ a torus bundle of relative dimension $\dim \Delta_S$ (cf. Lemma 2.1 and Proposition 1.3). Let \mathcal{A} be the Mumford model associated with $\overline{\mathcal{C}}$. By virtue of [17, Proposition 5.14], we have a unique morphism $\varphi' : \mathcal{X}'' \rightarrow \mathcal{A}$ extending f . Let us take a representative $w \in \mathbb{R}^n$ of \overline{w} . Taking account that

$$\overline{f}_{\text{aff}}(u) = \overline{\text{val}}(f(u)) = \overline{\text{val}}(\xi_W) = \overline{w} \in \text{relin}\{\overline{w}\},$$

we obtain a commutative diagram

$$\begin{array}{ccc} R & \xrightarrow{\tilde{\varphi}'} & Z_{\overline{w}} \\ \downarrow \tilde{\iota}' & & \downarrow \tilde{q}_w \\ S & \xrightarrow{\beta_w} & \tilde{\mathcal{B}}, \end{array}$$

which is moreover $(\mathbb{G}_m^r)_{\mathbb{K}}$ -equivariant, by Proposition 2.3.

The proof is given by an argument similar to that in Remark 3.1. Our assumption $f(u) = \xi_W$ together with Lemma 3.2 tells us that the generic point of R is mapped to that of W . Since $\text{in}_{\overline{w}}(X)$ is a closed subset of $Z_{\overline{w}}$, so is W , and hence W coincides with the closure of $\tilde{\varphi}'(R)$ in $Z_{\overline{w}}$. Accordingly, by the $(\mathbb{G}_m^r)_{\mathbb{K}}$ -equivariance of $\tilde{\varphi}' : R \rightarrow Z_{\overline{w}}$, we find that W is stable under the $(\mathbb{G}_m^r)_{\mathbb{K}}$ -action induced by \tilde{h}_{f, Δ_S} . Since \mathbb{T}'' is equal to the image of \tilde{h}_{f, Δ_S} , we see that W is stable under the action of \mathbb{T}'' . Since the action of \mathbb{T}'' on $Z_{\overline{w}}$ is free (cf. § 1.6) and $W \subset Z_{\overline{w}}$, the action on W is also free. Thus we obtain our first assertion.

Next let us prove the equivalence between (a), (b) and (c). Since Ξ is the quotient of W by the action of $(\mathbb{G}_m^r)_{\mathbb{K}}$ induced by $\tilde{h}_{\Delta_S, f} : (\mathbb{G}_m^r)_{\mathbb{K}} \rightarrow \mathbb{T}''$ and since $R/(\mathbb{G}_m^r)_{\mathbb{K}} = S$, the composite $R \rightarrow W \rightarrow \Xi$ factors through $R \rightarrow S$. Putting $\Upsilon := \tilde{q}_w(W)$, we then have a commutative diagram

$$\begin{array}{ccccccc} R & \xrightarrow{\tilde{\varphi}'} & W & \xrightarrow{\subset} & (\tilde{q}_w)^{-1}(\Upsilon) & \xrightarrow{\subset} & Z_{\overline{w}} \\ \tilde{\iota}' \downarrow & & \downarrow & & \downarrow & & \downarrow \tilde{q}_w \\ S & \longrightarrow & \Xi & \longrightarrow & \Upsilon & \xrightarrow{\subset} & \tilde{\mathcal{B}}. \end{array}$$

We have $\dim \Xi \geq \dim \Upsilon = b(W)$ since $\Xi \rightarrow \Upsilon$ is dominant. Taking into account the fact that the relative dimension of $W \rightarrow \Xi$ is equal to $\dim \mathbb{T}'' = \dim \overline{f}_{\text{aff}}(\Delta_S)$, we then have

$$(3.3.24) \quad \dim S = d - \dim \Delta_S \leq d - \dim \overline{f}_{\text{aff}}(\Delta_S) = \dim \Xi.$$

On the other hand, since $\tilde{\varphi}' : R \rightarrow W$ is dominant, we see that $S \rightarrow \Xi$ is dominant and hence $\dim S \geq \dim \Xi$. Consequently, we always have $\dim S = \dim \Xi \geq b(W)$, and conclude $\dim \Delta_S = \dim \overline{f}_{\text{aff}}(\Delta_S)$ by (3.3.24) in the setting of this proposition.

Now the equivalence between (b) and (c) is obvious. To show the equivalence between (a) and (b), we only have to show that $\dim S = \dim \beta_w(S)$ if and only if $\dim \Xi = b(W)$. However, it is also obvious since we know that $\dim S = \dim \Xi$ and $\dim \beta_w(S) = \dim \tilde{q}_w(\tilde{\varphi}'(R)) = b(W)$. \square

Remark 3.4. We always have $\dim S \geq b(W)$ in the situation of Proposition 3.3, as mentioned in the above proof.

4. STRICT SUPPORTS OF CANONICAL MEASURES

In this section, let K always denote a subfield of \mathbb{K} such that the valuation of \mathbb{K} restricts to a discrete valuation on K , and we assume that K is complete with respect to this valuation. Note that the value group of K is a discrete subgroup of \mathbb{Q} . Let K° denote the ring of integers of K , which is a discrete valuation ring. For a variety X over K and a scheme \mathcal{X} finite type over K° , we let X^{an} and \mathcal{X}^{an} stand for $(X \times_{\text{Spec } K} \text{Spec } \mathbb{K})^{\text{an}}$ and $(\hat{\mathcal{X}} \times_{\text{Spf } K^\circ} \text{Spf } \mathbb{K}^\circ)^{\text{an}}$ respectively, which are analytic spaces over \mathbb{K} . Although we deal only with analytic spaces which can be defined over K in this section, it is enough for our latter applications (cf. § 5.1).

4.1. Semistable alterations. Let \mathcal{X} be a connected admissible formal scheme over \mathbb{K} . A morphism $\mathcal{X}' \rightarrow \mathcal{X}$ is called a *semistable alteration* for \mathcal{X} if \mathcal{X}' is a connected strictly semistable formal scheme and it is a proper surjective generically finite morphism. The purpose of this subsection is to give some remarks on the existence of semistable alteration for a model of a closed subvariety of an abelian variety.

Let A be an abelian variety over K . First we recall that the Raynaud extension over K° exists for A , replacing K by a finite extension in \mathbb{K} if necessary (cf. [7, Theorem 1.1]). Then, we have an exact sequence

$$1 \longrightarrow \hat{\mathcal{T}}^\circ \longrightarrow \hat{\mathcal{A}}^\circ \longrightarrow \hat{\mathcal{B}} \longrightarrow 0$$

of formal group schemes over K° , where $\hat{\mathcal{T}}^\circ$ is a formal torus over K° , $\hat{\mathcal{A}}^\circ$ is the formal completion of a semiabelian scheme over K° , and $\hat{\mathcal{B}}$ is the formal completion of an abelian scheme over K° . Moreover, the base-change to \mathbb{K}° of the above sequence is nothing but the exact sequence (1.3.4) for $A \times_{\text{Spec } K} \text{Spec } \mathbb{K}$.

Let \mathcal{C} be a Λ -periodic rational¹⁰ polytopal decomposition of \mathbb{R}^n and let $\overline{\mathcal{C}}$ denote the induced polytopal decomposition of \mathbb{R}^n/Λ . Replacing K by a finite extension in \mathbb{K} if necessary again, we may assume that the coordinates of any vertex in \mathcal{C} sit in the value group of K . Then the Mumford model \mathcal{A} associated to $\overline{\mathcal{C}}$ can be defined over K° , i.e., there exists a formal model \mathcal{A} of A over K° such that $\mathcal{A} = \mathcal{A} \times_{\text{Spf } K^\circ} \text{Spf } \mathbb{K}^\circ$, because the construction of the Mumford models in [17, 4.7] can work over K° without any change. In fact, the polytopal domain U_Δ in [17, 4.5] can be defined over K by our assumption on \mathcal{C} , and we can take an affinoid atlas \mathcal{I} of B given by

$$\mathfrak{T} = \{\mathcal{V}^{\text{an}} \mid \mathcal{V} \text{ is a formal affine open subscheme of } \hat{\mathcal{B}}\},$$

where each $V \in \mathfrak{T}$ is defined over K . Therefore each $U_{V,\Delta}$ in [17, (7)] is defined over K , and hence we can make a formal analytic variety over K with the formal atlas $\{U_{V,\Delta}\}$ above by the same way as Gubler did in [17, 4.7]. Let \mathcal{A} be the formal scheme over K° corresponding

¹⁰We simply say “rational” for “ \mathbb{Q} -rational”.

this formal analytic variety. By the construction, we see that $\mathcal{A} \times_{\mathrm{Spf} K^\circ} \mathrm{Spf} \mathbb{K}^\circ$ coincides with the Mumford model \mathcal{A} over \mathbb{K}° recalled in § 1.6.

Let A be an abelian variety over \mathbb{K} and let X be a closed subvariety of A . Suppose that both of them can be defined over K . For a rational polytopal decomposition $\overline{\mathcal{C}_0}$, the closure \mathcal{X}_0 of X in the associated Mumford model \mathcal{A}_0 can be defined over K° . Then [13, Proposition 10.5] gives us a projective scheme $\mathcal{X}_1 \rightarrow \mathrm{Spec} K^\circ$ with generic fiber X , and a dominating morphism

$$\mathcal{X}_1 := \hat{\mathcal{X}}_1 \times_{\mathrm{Spf} K^\circ} \mathrm{Spf} \mathbb{K}^\circ \rightarrow \mathcal{X}_0$$

extending the identity on the generic fiber, where $\hat{\mathcal{X}}_1$ is the formal completion of \mathcal{X}_1 along its special fiber. In fact, the statement of [13, Proposition 10.5] is given under the assumption that the base field K is algebraically closed, but the proof delivered there works well also in our situation without any change. Applying [11, Theorem 6.5] to \mathcal{X}_1 and take the completion and the base-change to \mathbb{K} , we obtain a semistable alteration $\mathcal{X}' \rightarrow \mathcal{X}_1$. Consequently, we find that there is a semistable alteration $\mathcal{X}' \rightarrow \mathcal{X}_0$ by composition.

Remark 4.1. The semistable alteration $\mathcal{X}' \rightarrow \mathcal{X}_0$ constructed just above has a property that the generic fiber $(\mathcal{X}')^{\mathrm{an}} \rightarrow X^{\mathrm{an}}$ is the analytification of a morphism between projective varieties defined over a finite extension of K . In the sequel, we consider semistable alterations which satisfy this condition only.

4.2. Canonical measures and the canonical subset. Let X be a proper variety over K , and let L be a line bundle on X . Gubler defined the notion of *admissible metric on L* (cf. [17, 3.5]). If \overline{L} is a line bundle on X endowed with an admissible metric, then one can define a regular Borel measure $c_1(\overline{L})^{\wedge d}$ on X^{an} with suitable properties ([17, Proposition 3.8]). It was originally introduced by Chambert-Loir in [9]. Here we recall one important property of these measures: if $f : X' \rightarrow X$ is a morphism of d -dimensional geometrically integral proper varieties over K , then $f^*\overline{L}$ is an admissibly metrized line bundle on X' and

$$f_*(c_1(f^*\overline{L})^{\wedge d}) = \deg(f)c_1(\overline{L})^{\wedge d}.$$

Let A be an abelian variety over K and let L be an ample line bundle on A . As mentioned in [17, Example 3.7], we have an important metric, called *canonical metric*, on L , and the canonical metric is admissible. In the sequel, let \overline{L} always denote a line bundle endowed with a canonical metric for a line bundle on an abelian variety. For an irreducible closed subvariety X of A of dimension d , the restriction $\overline{L}|_X$ is a line bundle on X with an admissible metric (cf. [17, Proposition 3.6]). We can therefore define a *canonical measure*

$$\mu_{X^{\mathrm{an}}, L} := \frac{1}{\deg_L X} c_1(\overline{L})^{\wedge d},$$

which is a probability measure.

Let \mathcal{X} be a model of $X \subset A$ in a Mumford model of A , and let $\mathcal{X}' \rightarrow \mathcal{X}$ be a semistable alteration satisfying the assumption in Remark 4.1. As we saw in § 4.1, there always exists such a semistable alteration in our situation. Put $X' := (\mathcal{X}')^{\mathrm{an}}$ and let $f : X' \rightarrow A^{\mathrm{an}}$ be the composite $(\mathcal{X}')^{\mathrm{an}} \rightarrow X^{\mathrm{an}} \rightarrow A^{\mathrm{an}}$ as before. Then we have a measure

$$\mu_{X', f^*\overline{L}} := \frac{1}{\deg_{f^*L} X'} c_1(f^*\overline{L})^{\wedge d}$$

on X' , and we have

$$(4.1.25) \quad f_* \mu_{X', f^* \bar{L}} = \mu_{X^{\text{an}}, L}$$

as we recalled above. [17, Corollary 6.9] tells us that

$$(4.1.26) \quad \mu_{X', f^* \bar{L}} = \sum_{\Delta_S} r_S \delta_{\Delta_S},$$

where Δ_S runs through the set of non-degenerate canonical simplices, δ_{Δ_S} is the Lebesgue measure on Δ_S , and all coefficients r_S are positive. In particular $\mu_{X', f^* \bar{L}}$ is supported by $S(\mathcal{X}')_{nd-f}$, the union of the non-degenerate canonical simplices of $S(\mathcal{X}')$ with respect to f .

Since the non-degeneracy of Δ_S does not depend on L , the support of $\mu_{X', f^* \bar{L}}$ does not depend on L . Therefore the support of $\mu_{X^{\text{an}}, L}$ is exactly the image of $S(\mathcal{X}')_{nd-f}$, which implies that the support of $\mu_{X^{\text{an}}, L}$ depends only on X . Gubler called the support of $\mu_{X^{\text{an}}, L}$ the *canonical subset* of X^{an} in [17, Remark 6.11], which we denote by $S_{X^{\text{an}}}$ in this article. According to [17, Theorem 6.12], $S_{X^{\text{an}}}$ has a canonical rational piecewise linear structure. It is characterized by the property that for any model \mathcal{X} of X in a Mumford model and for any semistable alteration $\varphi : \mathcal{X}' \rightarrow \mathcal{X}$, the induced map $(\varphi)^{\text{an}}|_{S(\mathcal{X}')_{nd-f}} : S(\mathcal{X}')_{nd-f} \rightarrow S_{X^{\text{an}}}$ is a finite rational piecewise linear map (cf. [17, Theorem 6.12]).

Since $S_{X^{\text{an}}}$ is a rational piecewise linear space, we can consider a rational polytopal decomposition Σ of $S_{X^{\text{an}}}$. Recall that $\overline{\text{val}} : A^{\text{an}} \rightarrow \mathbb{R}^n/\Lambda$ denotes the valuation map. A polytopal decomposition Σ is said to be *descendible* if $\overline{\text{val}}_*(\Sigma) := \{\overline{\text{val}}(\sigma) | \sigma \in \Sigma\}$ is a polytopal decomposition of $\overline{\text{val}}(X^{\text{an}})$ and if $\overline{\text{val}}|_{\sigma} : \sigma \rightarrow \overline{\text{val}}(\sigma)$ is an isomorphism. We can see that for any rational polytopal decomposition Σ of $S_{X^{\text{an}}}$, there exists a rational subdivision Σ' of Σ which is descendible. To see that, we first note that $\overline{\text{val}}$ restricts to a surjective piecewise linear map $\overline{\text{val}}|_{S_{X^{\text{an}}}} : S_{X^{\text{an}}} \rightarrow \overline{\text{val}}(X^{\text{an}})$ with finite fiber since so is $\overline{\text{val}} \circ f : S(\mathcal{X}')_{nd-f} \rightarrow \overline{\text{val}}(X^{\text{an}})$. Accordingly, for each polytope $\sigma \in \Sigma$, the image $\overline{\text{val}}(\sigma)$ is a polyhedron, which implies that there exists a polytopal decomposition $\overline{\Sigma}'$ of $\overline{\text{val}}(X^{\text{an}})$ such that $\overline{\text{val}}(\sigma)$, for any $\sigma \in \Sigma$, is a finite union of polytopes in $\overline{\Sigma}'$. Then, for any $\overline{\sigma}' \in \overline{\Sigma}'$, $\overline{\text{val}}$ restricts to an isomorphism from each connected component of $\overline{\text{val}}^{-1}(\text{relin}(\overline{\sigma}'))$ to $\text{relin}(\overline{\sigma}')$. Collecting the closures of all connected components of $\overline{\text{val}}^{-1}(\text{relin}(\overline{\sigma}'))$ for all $\overline{\sigma}' \in \overline{\Sigma}'$, we can obtain a polytopal decomposition Σ' of $S_{X^{\text{an}}}$ such that $\overline{\text{val}}(\sigma') \in \overline{\Sigma}'$ for any $\sigma' \in \Sigma'$. Thus we obtain a subdivision of Σ which is descendible.

4.3. Strict supports of canonical measures. In this subsection, we define the notion of strict supports, and investigate the strict supports of a canonical measure on the canonical subset.

Definition 4.2. Let \mathcal{P} be a polytopal set with a finite polytopal decomposition Σ , and let μ be a semipositive Borel measure on \mathcal{P} . We say that $\sigma \in \Sigma$ is a *strict support* of μ if there exists an $\epsilon > 0$ such that $\mu - \epsilon \delta_{\sigma}$ is semipositive, where δ_{σ} is the relative Lebesgue measure on σ .

Let \mathcal{A}_0 be the Mumford model of A associated to a rational polytopal decomposition $\overline{\mathcal{C}}_0$ of \mathbb{R}^n/Λ and let \mathcal{X}_0 be the closure of X in \mathcal{A}_0 . Let $\varphi_0 : \mathcal{X}' \rightarrow \mathcal{A}_0$ be the composite of a semistable alteration $\mathcal{X}' \rightarrow \mathcal{X}_0$ with $\mathcal{X}_0 \hookrightarrow \mathcal{A}_0$, and put $f := (\varphi_0)^{\text{an}} : X' \rightarrow A^{\text{an}}$. A

polytopal decomposition Σ of $S_{X^{\text{an}}}$ is said to be φ_0 -subdivisional if for any non-degenerate simplex Δ_S of $S(\mathcal{X}')$, the image $f(\Delta_S)$ is a finite union of polytopes in Σ . There exists a φ_0 -subdivisional rational polytopal decomposition for any φ_0 since $f|_{\Delta_S} : \Delta_S \rightarrow S_{X^{\text{an}}}$ is a piecewise linear map for any non-degenerate Δ_S .

We give a remark on the structure of canonical measures here. Let $\mu_{X^{\text{an}},L}$ be the canonical measure on X^{an} , and let Σ be a descendible φ_0 -subdivisional rational polytopal decomposition of $S_{X^{\text{an}}}$. Then, by virtue of [17, Theorem 1.1, Theorem 6.7 and (39)], we can write

$$\overline{\text{val}}_*(\mu_{X^{\text{an}},L}) = \sum_{\sigma \in \Sigma} r_\sigma \delta_{\overline{\text{val}}(\sigma)},$$

where $r_\sigma \geq 0$ and $\delta_{\overline{\text{val}}(\sigma)}$ is the Lebesgue measure on $\overline{\text{val}}(\sigma)$. Since $\overline{\text{val}}$ is an affine map on each $\sigma \in \Sigma$, we can write

$$(4.2.27) \quad \mu_{X^{\text{an}},L} = \sum_{\sigma \in \Sigma} r'_\sigma \delta_\sigma,$$

where $r'_\sigma \geq 0$ and δ_σ is the Lebesgue measure on σ .

Lemma 4.3. *Let Σ be a descendible φ_0 -subdivisional rational polytopal decomposition of $S_{X^{\text{an}}}$, and take any $\sigma \in \Sigma$. Then, σ is a strict support of $\mu_{X^{\text{an}},L}$ if and only if there exists a non-degenerate canonical simplex Δ_S such that $\dim \Delta_S = \dim \sigma$ and $\sigma \subset f(\Delta_S)$.*

Proof. As we noted in the previous subsection, a canonical simplex Δ_S is a strict support of $\mu_{X',f^*\overline{L}}$ if and only if Δ_S is non-degenerate with respect to f (cf. (4.1.26)). Taking into account that Σ is φ_0 -subdivisional, we see from (4.1.25) that σ is a strict support of $\mu_{X^{\text{an}},L}$ if and only if there exists a non-degenerate Δ_S such that $\dim f(\Delta_S) = \dim \sigma$ and that $\sigma \subset f(\Delta_S)$. Moreover we have $\dim f(\Delta_S) = \dim \Delta_S$ since Δ_S is non-degenerate, and thus we obtain our assertion. \square

Let us fix an arbitrary rational point $\overline{w} \in \overline{\text{val}}(X^{\text{an}})$. For an irreducible component W of $\text{in}_{\overline{w}}(X)$, let ξ_W denote the point of X^{an} corresponding to the generic point of W . Note that $\overline{\text{val}}(\xi_W) = \overline{w}$.

Lemma 4.4. *Let Σ be a descendible φ_0 -subdivisional polytopal decomposition of $S_{X^{\text{an}}}$. Suppose that $\sigma \in \Sigma$ is a strict support of $\mu_{X^{\text{an}},L}$. Then for any rational point $t \in \text{relin } \sigma$ there exists a unique irreducible component W of $\text{in}_{\overline{\text{val}}(t)}(X)$ with $t = \xi_W$.*

Proof. The uniqueness is obvious since $W \neq W'$ implies $\xi_W \neq \xi_{W'}$. Let us prove the existence of W . Since Σ is φ_0 -subdivisional and σ is a strict support of $\mu_{X^{\text{an}},L}$, there exists a non-degenerate stratum Δ_S of $S(\mathcal{X}')$ such that $\dim \Delta_S = \dim \sigma$ and that $f(\Delta_S) \supset \sigma$ by Lemma 4.3. We can take a point $u \in \text{relin } \Delta_S$ with $f(u) = t$, and put $\overline{w} := \overline{\text{val}}(t)$.

Let $\overline{\mathcal{C}}$ be a rational subdivision of $\overline{\mathcal{C}}_0$ such that \overline{w} itself is a vertex in $\overline{\mathcal{C}}$. Let \mathcal{D} be a subdivision of $S(\mathcal{X}')$ given by

$$\mathcal{D} = \left\{ \Delta_S \cap (\overline{f}_{\text{aff}})^{-1}(\overline{\Delta}) \mid S \in \text{str}(\tilde{\mathcal{X}}'), \overline{\Delta} \in \overline{\mathcal{C}} \right\}.$$

Then u is a vertex of \mathcal{D} . Let \mathcal{A} be the Mumford model of A associated to $\overline{\mathcal{C}}$, \mathcal{X}'' the formal model associated to the subdivision \mathcal{D} and let R be the stratum of $\tilde{\mathcal{X}}''$ corresponding to u . Then we have an extension $\varphi' : \mathcal{X}'' \rightarrow \mathcal{A}$ of $f : X' \rightarrow A^{\text{an}}$. Let W be the closure of the

image $\tilde{\varphi}'(R)$ in $Z_{\overline{w}}$. Since $\tilde{\varphi}'(R) \subset \text{in}_{\overline{w}}(X)$ and $\text{in}_{\overline{w}}(X)$ is a closed subscheme of $Z_{\overline{w}}$, we see that $W \subset \text{in}_{\overline{w}}(X)$. Since Δ_S is non-degenerate, Remark 3.1 tells us that $\dim W = d$, where $d = \dim X$. Taking into account the dimension, we then conclude that W is an irreducible component of $\text{in}_{\overline{w}}(X)$. Since the point in $(\mathcal{X}')^{\text{an}}$ corresponding to the generic point of R maps to ξ_W , we obtain $t = f(u) = \xi_W$ by Lemma 3.2 as required. \square

We know that the image of a non-degenerate stratum in the canonical subset is a strict support of a canonical measure (cf. Lemma 4.3). Now we can show the following statement, which says that its converse also holds true:

Theorem 4.5. *Let L be an ample line bundle on A . Let Σ be a descendible φ_0 -subdivisional polytopal decomposition of $S_{X^{\text{an}}}$, and let $\sigma \in \Sigma$ and S be a stratum of $\tilde{\mathcal{X}}'$ such that $f(\Delta_S) \supset \sigma$ and $\dim \overline{f}_{\text{aff}}(\Delta_S) = \dim \overline{\text{val}}(\sigma)$. Then Δ_S is non-degenerate with respect to f if and only if σ is a strict support of $\mu_{X^{\text{an}}, L}$.*

Proof. The “only if” part is immediate from Lemma 4.3. Let us show the “if” part. Since Σ is φ_0 -divisional and σ is a strict support of $\mu_{X^{\text{an}}, L}$, we can take a non-degenerate canonical simplex $\Delta_{S'}$ of $S(\mathcal{X}')$ such that $\sigma \subset f(\Delta_{S'})$ and $\dim \sigma = \dim \Delta_{S'}$ by Lemma 4.3. Note that we have $\overline{\text{val}}(\sigma) \subset \overline{f}_{\text{aff}}(\Delta_S)$, $\overline{\text{val}}(\sigma) \subset \overline{f}_{\text{aff}}(\Delta_{S'})$ and $\dim \overline{\text{val}}(\sigma) = \dim \overline{f}_{\text{aff}}(\Delta_S) = \dim \overline{f}_{\text{aff}}(\Delta_{S'})$.

Take a rational point $t \in \text{relin } \sigma$ and put $\overline{w} := \overline{\text{val}}(t)$. Since σ is a strict support of $\mu_{X^{\text{an}}, L}$, there exists a unique irreducible component W of $\text{in}_{\overline{w}}(X)$ with $t = \xi_W$ by Lemma 4.4. Let \mathbb{T}'' be the torus parallel to $\overline{\text{val}}(\sigma)$. Since $\Delta_{S'}$ is non-degenerate with respect to f and \mathbb{T}'' is also the torus parallel to $\overline{f}_{\text{aff}}(\Delta_{S'})$, we have $\dim W/\mathbb{T}'' = b(W)$ by Proposition 3.3. Since \mathbb{T}'' is also the torus parallel to $\overline{f}_{\text{aff}}(\Delta_S)$, we conclude that Δ_S is non-degenerate with respect to f again by Proposition 3.3. \square

5. NON-DENSITY OF SMALL POINTS AND THE GEOMETRIC BOGOMOLOV CONJECTURE

5.1. Notations, convention and remarks. Let k be an algebraically closed field, \mathfrak{B} an irreducible normal projective variety over k , and let \mathcal{H} be an ample line bundle on \mathfrak{B} .¹¹ Let K be the function field of \mathfrak{B} , and let \overline{K} be an algebraic closure of K . All of them are fixed in the sequel.

For a finite extension K' of K , let $\mathfrak{B}_{K'}$ denote the normalization of \mathfrak{B} in K' . Let $M_{K'}$ denote the set of points in $\mathfrak{B}_{K'}$ of codimension one. For any $w \in M_{K'}$, the local ring $\mathcal{O}_{\mathfrak{B}_{K'}, w}$ is a discrete valuation ring with the fraction field K' , and the order function $\text{ord}_w : (K')^\times \rightarrow \mathbb{Z}$ gives an additive discrete valuation. If K'' is a finite extension of K' , then we have a canonical finite surjective morphism $\mathfrak{B}_{K''} \rightarrow \mathfrak{B}_{K'}$, which induces a surjective map $M_{K''} \rightarrow M_{K'}$. Thus we have an inverse system $(M_{K'})_{K'}$, where K' runs through the finite extensions of K in \overline{K} , and hence we define $M_{\overline{K}} := \varprojlim_{K'} M_{K'}$. We call an element of $M_{\overline{K}}$ a *place* of \overline{K} . Each place $v = (v_{K'})_{K'} \in M_{\overline{K}}$ determines a unique non-archimedean multiplicative value $|\cdot|_v$ on \overline{K} in such a way that the following conditions are satisfied.

- The restriction of $|\cdot|_v$ to K' is equivalent to the valuation associated with the order function $\text{ord}_{v_{K'}}$.
- For any $x \in K^\times$, $|x|_v = e^{-\text{ord}_{v_K} x}$.

¹¹ We assume \mathfrak{B} to be a curve in § 5.6.

Through this correspondence, we regard a place of \overline{K} as a valuation of \overline{K} . For a $v \in M_{\overline{K}}$, let \overline{K}_v denote the completion of \overline{K} with respect to v . It is an algebraically closed field complete with respect to the non-archimedean valuation $|\cdot|_v$.

For each $v_K \in M_K$, let $|\cdot|_{v_K, \mathcal{H}}$ be the valuation normalized in such a way that

$$|x|_{v_K, \mathcal{H}} := e^{-(\text{ord}_{v_K} x)(\deg_{\mathcal{H}} v_K)},$$

where $\deg_{\mathcal{H}} v_K$ stands for the degree of the closure of v_K in \mathfrak{B} with respect to \mathcal{H} . It is well known that the set $\mathfrak{V} := \{|\cdot|_{v_K, \mathcal{H}}\}_{v \in M_K}$ of valuations satisfies the product formula, and hence we can define the notion of heights with respect to this set of valuations, namely, an absolute logarithmic height with respect to \mathfrak{V} (cf. [19, Chapter 3 § 3]). The “height” in this article always means this height.

Let F/k be any field extension. For a scheme X over k , we write $X_F := X \times_{\text{Spec } k} \text{Spec } F$. If $\phi : X \rightarrow Y$ is a morphism of schemes over k , we write $\phi_F : X_F \rightarrow Y_F$ for the base extension to F .

Let X be an algebraic scheme over \overline{K} . For each place v of \overline{K} , we have a Berkovich analytic space associated to $X \times_{\text{Spec } \overline{K}} \text{Spec } \overline{K}_v$. We write X_v for this analytic space. Let A be an abelian variety defined over \overline{K} and suppose that X is a subvariety of A . Then A and X can be defined over a finite extension of K in \overline{K} , and hence A_v and X_v can be defined over a valuation subfield of \overline{K}_v of which valuation is a discrete valuation. Therefore, the assumptions in § 4 are fulfilled for them, and we can apply the arguments and the results in § 4 in this setting.

5.2. Geometric Bogomolov conjecture. Let L be an even ample line bundle on an abelian variety A over \overline{K} , where “even” means $[-1]^*L = L$. We have the canonical height function $\hat{h}_L : A(\overline{K}) \rightarrow \mathbb{R}$, which is a non-negative function.

Let X be an irreducible closed subvariety of A . For a positive number ϵ , we put

$$X(\epsilon; L) := \{x \in X(\overline{K}) \mid \hat{h}_L(x) < \epsilon\}.$$

We say X has *dense small points* if $X(\epsilon; L)$ is Zariski dense in X . Recall that this notion does not depend on the choice of an even ample line bundle L (cf. [29, Definition 2.2]). We define the *stabilizer* of X by

$$(5.0.28) \quad G_X := \{a \in A \mid a + X \subset X\}.$$

It is a closed subgroup of A . Next we recall the notion of special subvarieties defined in [29]. Let $(A^{\overline{K}/k}, \text{Tr}_A)$ be the \overline{K}/k -trace of A , that is, $A^{\overline{K}/k}$ is an abelian variety over k and $\text{Tr}_A : (A^{\overline{K}/k})_{\overline{K}} \rightarrow A$ is a homomorphism, and they satisfy the following universal property: for any abelian variety A' over k and for any homomorphism $\phi : A'_{\overline{K}} \rightarrow A$, there exists a unique homomorphism $\text{Tr}(\phi) : A' \rightarrow A^{\overline{K}/k}$ over k such that $\text{Tr}_A \circ \text{Tr}(\phi)_{\overline{K}} = \phi$. We refer to [18] and [19] for details. Then we say X is *special* if there exist a torsion point $\tau \in A(\overline{K})$ and a closed subvariety $X' \subset X^{\overline{K}/k}$ over k such that

$$X = G_X + \text{Tr}_A(X'_{\overline{K}}) + \tau.$$

We say a point $x \in A$ is a *special point* if $\{x\}$ is a special subvariety. In the definition of special subvarieties above, we can replace the condition for τ being torsion by that for τ

being special (cf. [29, Remark 2.6]). Note also that X is a special subvariety of A if and only if X/G_X is a special subvariety of A/G_X (cf. [29, Proposition 2.11]).

Remark 5.1. We can see that X is a special subvariety if and only if there exist an abelian variety C over k , a closed subvariety $X' \subset C$, a homomorphism $\phi : C_{\overline{K}} \rightarrow A$, and a torsion point $\tau \in A(\overline{K})$ such that $X = G_X + \phi(X'_{\overline{K}}) + \tau$. In fact, the “only if” part is trivial, and “if” part is also easy since we have a homomorphism $\text{Tr}(\phi) : C \rightarrow A^{\overline{K}/k}$ such that $\phi = \text{Tr}_A \circ \text{Tr}(\phi)_{\overline{K}}$ by the universality of the trace.

A point is special if and only if it is of height zero (cf. [29, (2.5.4)]), and hence $\{x\}$ has dense small points if and only if x is a special point. In other word, for an irreducible subvariety of *dimension zero*, being special is the same thing as having dense small points. Even in the case of positive dimension, it is not difficult to see that any special subvariety has dense small points (cf. [29, Corollary 2.8]), but we do not know whether the converse holds true or not in general. The geometric Bogomolov conjecture insists that it should hold true:

Conjecture 5.2 (Geometric Bogomolov conjecture for abelian varieties). Let A be an abelian variety over \overline{K} and let $X \subset A$ be an irreducible closed subvariety. Then X should not have dense small points unless X is a special subvariety.

Although there are some partial answers (cf. [15, 29]), this conjecture is still open in full generality. In the rest of this paper, we will apply Theorem 4.5 to show that the geometric Bogomolov conjecture holds for a large class of abelian varieties including the cases of [15] and [29] (cf. Corollary 5.14), and we will also show that the geometric Bogomolov conjecture for all abelian varieties can be reduced to that for abelian varieties without degeneration (cf. Theorem 6.3 and Corollary 6.4).

5.3. Tropically trivial subvarieties and density of small points. We begin this subsection with a definition:

Definition 5.3. Let A be an abelian variety over \overline{K} and let X be an irreducible closed subvariety of A . We say X is *tropically trivial* if $\overline{\text{val}}(X_v)$ consists of a single point for any $v \in M_{\overline{K}}$, where $\overline{\text{val}}$ is the valuation map for A_v .

From the viewpoint of the geometric Bogomolov conjecture, it is interesting to ask when a closed subvariety of an abelian variety does not have dense small points. The following theorem is an answer to it, which tells us that if a subvariety, after divided by its stabilizer, is tropically non-trivial at some place, then it should not have dense small points:

Theorem 5.4. Let A be an abelian variety over \overline{K} and let X be a closed subvariety of A . If X has dense small points, then the closed subvariety X/G_X of A/G_X is tropically trivial.

Proof. Suppose that we have a counterexample X : A certain subvariety X has dense small points but there exists a place $v \in M_{\overline{K}}$ such that $\overline{\text{val}}((X/G_X)_v)$ is not one point. Then we may assume that the stabilizer G_X is trivial since X/G_X is also a counterexample by [29, Lemma 2.1].

We consider a homomorphism $\alpha : A^N \rightarrow A^{N-1}$ defined by $(x_1, \dots, x_N) \mapsto (x_2 - x_1, \dots, x_N - x_{N-1})$. Put $Z := X^N \subset A^N$ and $Y := \alpha(Z)$. Since $G_X = 0$, the restriction $Z \rightarrow Y$ of α is a

generically finite surjective morphism for a large N (cf. [31, Lemma 3.1]), and we fix such an N . Let the same symbol α denote the morphism $Z_v \rightarrow Y_v$ between associated analytic spaces over \overline{K}_v .

Let \hat{h}_{A^N} and $\hat{h}_{A^{N-1}}$ be the canonical height function associated to even ample line bundles on A^N and A^{N-1} respectively. Since X has dense small points, so does Z (cf. [29, Lemma 2.4]), and we can find a generic net $(P_m)_{m \in I}$, where I is a directed set, such that $\lim_m \hat{h}_Z(P_m) = 0$ (cf. [15, Proof of Theorem 1.1]). The image $(\alpha(P_m))_{m \in I}$ is also a generic net of Y with $\lim_m \hat{h}_Y(\alpha(P_m)) = 0$. Let K' be a finite extension of K in \overline{K} over which A and X , and hence Z and Y are all defined. For a point P in $Z(\overline{K})$ or in $Y(\overline{K})$, let $O(P)$ be the $\text{Gal}(\overline{K}/K')$ -orbit of P . Then by the equidistribution theorem [15, Theorem 1.1], we find

$$\nu_{Z_v, m} := \frac{1}{|O(P_m)|} \sum_{z \in O(P_m)} \delta_z \quad \text{and} \quad \nu_{Y_v, m} := \frac{1}{|O(\alpha(P_m))|} \sum_{y \in O(\alpha(P_m))} \delta_y$$

weakly converge, as $m \rightarrow \infty$, to the canonical measures μ_{Z_v} on Z_v and to μ_{Y_v} on Y_v associated to even ample line bundles respectively. Since $\alpha_*(\nu_{Z_v, m}) = \nu_{Y_v, m}$, we obtain $\alpha_*(\mu_{Z_v}) = \mu_{Y_v}$. Note in particular that $\alpha(S_{Z_v}) = S_{Y_v}$, where S_{Z_v} and S_{Y_v} are the canonical subsets.

We take Mumford models of A_v^{N-1} and of A_v^N respectively so that the map α extends to a morphism between the Mumford models. Let \mathcal{Y} and \mathcal{Z} be the closure of Y_v and Z_v in these Mumford models respectively. Let $\varphi : \mathcal{Z}' \rightarrow \mathcal{Z}$ be a semistable alteration for \mathcal{Z} as in § 4.1. Note that the composite $\psi : \mathcal{Z}' \rightarrow \mathcal{Z} \rightarrow \mathcal{Y}$ is a semistable alteration for \mathcal{Y} . Let g be the composite $(\mathcal{Z}')^{\text{an}} \rightarrow Z_v \hookrightarrow A_v^{N-1}$ and put $h := \alpha \circ g : (\mathcal{Z}')^{\text{an}} \rightarrow A_v^N$.

Let us choose a descendible φ -subdivisional rational polytopal structure Σ_{Z_v} of S_{Z_v} and a descendible ψ -subdivisional one Σ_{Y_v} of S_{Y_v} . Taking a subdivision of Σ_{Z_v} if necessary, we may assume that for any $\sigma' \in \Sigma_{Z_v}$, there exists a unique $\sigma'' \in \Sigma_{Y_v}$ such that $\text{relin}(\alpha(\sigma')) \subset \text{relin}(\sigma'')$. Since $\overline{\text{val}}(X_v)$ contains a polytope of positive dimension by our assumption, and since $\overline{\text{val}}(X_v)$ coincides with the support of $\overline{\text{val}}_*(\mu_{X_v})$ by [17, Theorem 1.1], there exists a positive dimensional polytope P of $\overline{\text{val}}(X_v)$ such that $\overline{\text{val}}_*(\mu_{X_v}) - \epsilon \delta_P$ is semipositive for a small $\epsilon > 0$. Using [29, Lemma 4.1 and Proposition 4.5], we see that $\overline{\text{val}}_*(\mu_{Z_v}) - \epsilon \delta_{P^N}$ is semipositive for a small $\epsilon > 0$. It follows that there is a polytope $\sigma \in \Sigma_{Z_v}$ such that $\text{relin}(\overline{\text{val}}(\sigma)) \cap \text{relin}(P^N) \neq \emptyset$, $\dim \overline{\text{val}}(\sigma) = \dim P^N$, and that σ is a strict support of μ_{Z_v} . Since $\overline{\alpha}_{\text{aff}}$ is an affine map and it contracts the diagonal of P^N to a point, we have $\dim \overline{\alpha}_{\text{aff}}(P^N) < \dim P^N$ and hence $\dim \overline{\text{val}}(\alpha(\sigma)) < \dim \overline{\text{val}}(\sigma)$. This inequality concludes

$$(5.4.29) \quad \dim \alpha(\sigma) < \dim \sigma$$

since the dimension of a polytope in the canonical subset coincides with that of its tropicalization.

Let $\tau \in \Sigma_{Y_v}$ be a polytope such that $\text{relin}(\alpha(\sigma)) \subset \text{relin} \tau$. By our assumption on Σ_{Z_v} and Σ_{Y_v} made above, such a polytope τ uniquely exists and is characterized by the condition that $\text{relin}(\alpha(\sigma)) \cap \text{relin} \tau \neq \emptyset$. We claim here that $\dim \tau = \dim \alpha(\sigma)$ and τ is a strict support of μ_{Y_v} . Take a non-empty compact subset $V \subset \text{relin}(\alpha(\sigma))$ which is the closure of an open subset of $\alpha(\sigma)$. Then there is an $\epsilon' > 0$ such that $\alpha_* \delta_\sigma - \epsilon' \delta_V \geq 0$. Since σ is a strict support of μ_{Z_v} , there exists an $\epsilon'' > 0$ such that $\mu_{Z_v} - \epsilon'' \delta_\sigma \geq 0$. Accordingly, putting $\epsilon := \epsilon' \epsilon''$, we obtain

$$\mu_{Y_v} - \epsilon \delta_V \geq \alpha_* \mu_{Z_v} - \epsilon'' \alpha_* \delta_\sigma = \alpha_*(\mu_{Z_v} - \epsilon'' \delta_\sigma) \geq 0.$$

On the other hand, recall that we can write

$$\mu_{Y_v} = \sum_{\sigma' \in \Sigma_{Y_v}} r_{\sigma'} \delta_{\sigma'}, \quad r_{\sigma'} \geq 0$$

(cf. (4.2.27)). Therefore we can find a polytope $\tau' \in \Sigma_{Y_v}$ such that $V \subset \tau'$, $\dim V = \dim \alpha(\sigma) = \dim \tau'$ and such that τ' is a strict support of μ_{Y_v} . Since τ is the unique polytope in Σ_{Y_v} with $V \subset \text{relin } \tau$, we conclude that $\tau = \tau'$ and hence τ is a strict support of μ_{Y_v} with $\dim \tau = \dim V = \dim \alpha(\sigma)$.

Since σ is a strict support of μ_{Z_v} , Lemma 4.3 gives us a canonical simplex Δ_S of $S(\mathcal{Z})$, non-degenerate with respect to g , such that $g(\Delta_S) \supset \sigma$ and $\dim \Delta_S = \dim \sigma$. We have

$$\dim \bar{h}_{\text{aff}}(\Delta_S) = \dim \bar{\alpha}_{\text{aff}}(\bar{g}_{\text{aff}}(\Delta_S)) = \dim \bar{\alpha}_{\text{aff}}(\overline{\text{val}}(\sigma)) = \dim \overline{\text{val}}(\alpha(\sigma)) = \dim \alpha(\sigma) = \dim \tau,$$

and hence the polytopes $h(\Delta_S)$, τ and $\alpha(\sigma)$ have the same dimension. Since both $h(\Delta_S)$ and τ contain $\alpha(\sigma)$ and Σ_{Y_v} is ψ -subdivisional, we find $h(\Delta_S) \supset \tau$. Since τ is a strict support of μ_{Y_v} , Δ_S must be non-degenerate with respect to h by virtue of Theorem 4.5. On the other hand, we have

$$\dim \bar{h}_{\text{aff}}(\Delta_S) = \dim \alpha(\sigma) < \dim \sigma = \dim \Delta_S$$

by (5.4.29). That in particular says that Δ_S should be degenerate with respect to h . That is a contradiction and thus we obtain our theorem. \square

The following assertion follows immediately since a special subvariety has dense small points:

Corollary 5.5. *If X is a special subvariety of A , then X/G_X is tropically trivial.*

Let us give a remark. The assertion of Corollary 5.5 itself can be shown directly, though we established it as a corollary of Theorem 5.4. In fact, after taking the quotient by G_X , we may assume $G_X = 0$ and $0 \in X$. Since X is a special subvariety, we can take an abelian variety A' over k , a closed subvariety $Y \subset A'$ and a homomorphism $\alpha : A'_{\overline{K}} \rightarrow A$ such that $\alpha(Y_{\overline{K}}) = X$, after translating X by a special point if necessary. Since $A'_{\overline{K}}$ is nowhere degenerate, the subvariety $Y_{\overline{K}}$ is tropically trivial, and hence its image X is also tropically trivial (cf. (1.3.11)).

5.4. The nowhere-degeneracy rank. In this subsection, we define the notion of nowhere-degeneracy rank of an abelian variety, and show some properties on it.

An abelian variety A over \overline{K} is said to be *somewhere degenerate* if A is degenerate at some place of \overline{K} , and is said to be *nowhere degenerate* if A_v is non-degenerate for all $v \in M_{\overline{K}}$.

Lemma 5.6. *Let A and A' be abelian varieties which are trivial or isogenous to the product of somewhere degenerate simple abelian varieties,¹² and let B and B' be nowhere degenerate abelian varieties. Then, for any homomorphism $\phi : A \times B \rightarrow A' \times B'$, there exist homomorphisms $\phi' : A \rightarrow A'$ and $\phi'' : B \rightarrow B'$ such that $\phi = \phi' \times \phi''$.*

Proof. It suffices to show that any homomorphisms $\psi : A \rightarrow B'$ and $\psi' : B \rightarrow A'$ are trivial. We first show the following claim, which is a special case of this assertion.

¹²With the notion of non-degeneracy rank, this condition can be stated as $\text{nd-rk}(A) = \text{nd-rk}(A') = 0$ (cf. Remark 5.8).

Claim 5.6.30. Let A be a somewhere degenerate simple abelian variety and let B be a nowhere degenerate abelian variety. Then any homomorphism $\phi : A \rightarrow B$ and any homomorphism $\phi' : B \rightarrow A$ are trivial.

Proof. Let v be a place at which A is degenerate. We consider a homomorphism $\phi : A \rightarrow B$ first, and suppose that it is not trivial. Since A is simple, ϕ must be finite, and we have an exact sequence

$$0 \rightarrow (A/\text{Ker } \phi)_v \rightarrow B_v \rightarrow (\text{Coker } \phi)_v \rightarrow 0.$$

Note that $(A/\text{Ker } \phi)_v$ is degenerate at v by Lemma 1.4. On the other hand, since B_v is non-degenerate, Proposition 1.6 shows us that $(A/\text{Ker } \phi)_v$ should be non-degenerate. That is a contradiction, and thus we see that ϕ is trivial.

Next let us consider ϕ' . If it is not trivial, then ϕ' must be surjective since A is simple, and we obtain an exact sequence

$$0 \rightarrow (\text{Ker } \psi)_v \rightarrow B_v \rightarrow A_v \rightarrow 0.$$

Then using Proposition 1.6 again, we obtain a contradiction similarly. \square

Let us show that $\psi : A \rightarrow B'$ is trivial first. We may assume A to be non-trivial. We take somewhere degenerate simple abelian varieties A_1, \dots, A_s ($s \geq 1$) and an isogeny $\alpha : A_1 \times \dots \times A_s \rightarrow A$. We define a homomorphism $\psi_i : A_i \rightarrow B'$ as the composite

$$A_i \hookrightarrow A_1 \times \dots \times A_s \rightarrow A \rightarrow B'$$

of the canonical injective homomorphism $A_i \hookrightarrow A_1 \times \dots \times A_s$, α , and ψ . Then ψ_i is trivial by Claim 5.6.30 for all $i = 1, \dots, s$, which tells us that $\psi \circ \alpha$ is trivial. Since α is surjective, we conclude that ψ is trivial.

In order to show that $\psi' : B \rightarrow A'$ is trivial, we may assume A' to be non-trivial as well, and take simple abelian varieties $A'_1, \dots, A'_{s'}$ ($s' \geq 1$) and an isogeny $\alpha' : A' \rightarrow A'_1 \times \dots \times A'_{s'}$. Let $p_i : A'_1 \times \dots \times A'_{s'} \rightarrow A'_i$ denote the canonical projection. Then Claim 5.6.30 tells us that the homomorphism $p_i \circ \alpha' \circ \psi'$ is trivial for any $i = 1, \dots, s'$, which implies that $\alpha' \circ \psi'$ is trivial. Since α' is finite, we conclude that ψ' is trivial. \square

It is well known that any abelian variety is isogenous to the product of simple abelian varieties (cf. [20]). Applying this fact to an abelian variety A over \overline{K} , we see that there exist an abelian variety A_* that is trivial or isogenous to the product of somewhere degenerate simple abelian varieties, and a nowhere degenerate abelian variety B , such that $A_* \times B$ is isogenous to A . If A'_* and B' are also such abelian varieties for A , then there exist isogenies $A_* \times B \rightarrow A$ and $A \rightarrow A'_* \times B'$. Applying Lemma 5.6 to the composite of these two isogenies, we obtain isogenies $A_* \rightarrow A'_*$ and $B \rightarrow B'$. It follows that the isogeny classes of such A_* and B above are well-defined for A .

Definition 5.7. With the notation above, the isogeny class of B is called the *nowhere degenerate factor* for A . The dimension of a representative B of the nowhere degenerate factor for A is called the *nowhere-degeneracy rank of A* , which is denoted by $\text{nd-rk}(A)$.

Let B be a representative of the nowhere degenerate factor for A . The above discussion shows that A is isogenous to $A_* \times B$ for some A_* , and we automatically have $\text{nd-rk}(A_*) = 0$.

Remark 5.8. For an abelian variety A over \overline{K} , we have $\text{nd-rk}(A) = 0$ if and only if A is trivial or isogenous to the product of somewhere degenerate simple abelian varieties.

The following assertion is a corollary of Lemma 5.6.

Corollary 5.9. *Let $\phi : A \rightarrow A'$ be a surjective homomorphism of abelian varieties. Then $\text{nd-rk}(A) \geq \text{nd-rk}(A')$.*

Proof. Let $\alpha : A_* \times B \rightarrow A$ and $\alpha' : A' \rightarrow A'_* \times B'$ be isogenies of abelian varieties, where B and B' are representatives of the nowhere degenerate factors for A and A' respectively. Note that $\text{nd-rk}(A_*) = \text{nd-rk}(A'_*) = 0$. Apply Lemma 5.6 to $\alpha' \circ \phi \circ \alpha$, we obtain homomorphisms $\psi' : A_* \rightarrow A'_*$ and $\psi'' : B \rightarrow B'$ such that $\alpha' \circ \phi \circ \alpha = \psi' \times \psi''$. Since $\alpha' \circ \phi \circ \alpha$ is surjective, we find that ψ'' is also surjective, which shows us $\dim B \geq \dim B'$. Thus we obtain $\text{nd-rk } A \geq \text{nd-rk } A'$. \square

5.5. Proof of Theorem D. We give a proof of Theorem D, one of our main results, in this subsection.

Before proving the proof, we would like to give a condition for an irreducible closed subvariety to be tropically trivial. To do that, we show a property on the subgroup generated by a subvariety. For a subvariety X of an abelian variety A , let $\langle X \rangle$ denote the abelian subvariety of A generated by X , that is, the smallest abelian subvariety containing X .

Lemma 5.10. *Let X be an irreducible closed subvariety of an abelian variety A over \overline{K} , with $0 \in X$. Then $\langle X \rangle_v$ is the smallest analytic subgroup of A_v containing X_v .*

Proof. Let us consider, for an $l \in \mathbb{N}$, a morphism $X^{2l} \rightarrow A$ given by

$$(5.10.31) \quad (x_1, x_2, \dots, x_{2l-1}, x_{2l}) \mapsto (x_1 - x_2) + \dots + (x_{2l-1} - x_{2l}).$$

Let X_l be the image of this morphism. It is an irreducible closed subvariety of A . Taking account that $0 \in X$, we find

$$X \subset X_1 \subset X_2 \subset \dots \subset X_l \subset \dots.$$

Since each X_l is an irreducible closed subset, there exists l_0 such that $X_l = X_{l_0}$ for all $l \geq l_0$. We have in particular $\bigcup_{l \in \mathbb{N}} X_l = X_{l_0}$. On the other hand, we have $X_l + X_m \subset X_{l+m}$, $0 \in X_l$ and $-X_l = X_l$ for all $l, m \in \mathbb{N}$ by their definitions. That tells us that $\bigcup_{l \in \mathbb{N}} X_l = X_{l_0}$ is a subgroup scheme, and hence it is an abelian subvariety as X_{l_0} is an irreducible closed subvariety. Since $\langle X \rangle$ is an abelian subvariety containing X , we have $X_l \subset \langle X \rangle$ for all l . The minimality of $\langle X \rangle$ concludes $\langle X \rangle = X_{l_0}$.

Let B be an analytic subgroup of A_v containing X_v . We then have $B \supset (X_{l_0})_v$ by the definition of X_{l_0} , and hence $B \supset \langle X \rangle_v$. That implies that $\langle X \rangle_v$ is the smallest analytic subgroup containing X_v . \square

The following lemma will be used in the proof of Proposition 5.12.

Lemma 5.11. *Suppose that A is a simple abelian variety degenerate at a place v , and let X be an irreducible closed subvariety of A . Then X consists of a single point if so does $\overline{\text{val}}(X_v)$.*

Proof. Translating X by a point in $A(\overline{K})$ if necessary, we may assume that $0 \in X$. Then $\overline{\text{val}}(X_v) = \{\overline{0}\}$ and hence $X_v \subset \overline{\text{val}}^{-1}(\overline{0})$. Since $\overline{\text{val}}^{-1}(\overline{0})$ is an analytic subgroup of A_v , Lemma 5.10 tells us $\langle X \rangle_v \subsetneq A_v$ and hence $\langle X \rangle \subsetneq A$. Then we have $\langle X \rangle = \{0\}$ since A is simple, which shows our claim. \square

Proposition 5.12. *Let X be an irreducible closed subvariety of an abelian variety A over \overline{K} . Then the following statements are equivalent to each other.*

- (a) *X is tropically trivial.*
- (b) *Let A' be an abelian variety with $\text{nd-rk}(A') = 0$, B' a nowhere degenerate abelian variety, and let $\phi : A \rightarrow A' \times B'$ be a homomorphism. Then there exists a point $a' \in A'(\overline{K})$ such that $\phi(X) \subset \{a'\} \times B'$.*

Proof. We show that (a) implies (b) first. We may assume A' to be non-trivial since otherwise our assertion is trivial. Let $\phi : A \rightarrow A' \times B'$ be a homomorphism as in (b). Let $p : A' \times B' \rightarrow A'$ be the canonical projection and put $\psi := p \circ \phi$. It suffices to show that $\psi(X)$ consists of one point. Our assumption $\text{nd-rk}(A') = 0$ allows us to take an isogeny $\alpha : A' \rightarrow A'_1 \times \cdots \times A'_s$ such that each A'_i is a simple abelian variety which is degenerate at some place v_i . Let $p_i : A'_1 \times \cdots \times A'_s \rightarrow A'_i$ be the canonical projection for each i , and put $\psi_i := p_i \circ \alpha \circ \psi$. Since $\overline{\text{val}}(X_{v_i})$ is a point by our assumption, so is $\overline{\text{val}}(\psi_i(X)_{v_i}) = \overline{\psi_{i,\text{aff}}}(\overline{\text{val}}(X_{v_i}))$, where $\overline{\psi_{i,\text{aff}}}$ is the affine map defined in § 1.5 for ψ_i . Then by Lemma 5.11, we find that $\psi_i(X)$ is one point for all i , which implies that $\alpha \circ \psi(X)$ is one point. Since α is finite and $\psi(X)$ is connected, we conclude $\psi(X)$ is one point.

Next we show that (b) implies (a). There exists an isogeny $\phi : A \rightarrow A' \times B'$, where A' is an abelian variety with $\text{nd-rk}(A') = 0$ and B' is a representative of nowhere degenerate factor for A . Take an arbitrary $v \in M_{\overline{K}}$. Since $\phi(X) \subset \{a'\} \times B'$ for some $a' \in A'(\overline{K})$ and B' is nowhere degenerate, the set $\overline{\text{val}}(\phi(X_v))$ consists of a single point. Taking account that $\overline{\phi_{\text{aff}}}(\overline{\text{val}}(X_v)) = \overline{\text{val}}(\phi(X_v))$, we find that $\overline{\phi_{\text{aff}}}(\overline{\text{val}}(X_v))$ consists of a single point. Since $\overline{\phi_{\text{aff}}}$ is a finite map (cf. Remark 1.5) and $\overline{\text{val}}(X_v)$ is connected, we conclude that $\overline{\text{val}}(X_v)$ is one point. Thus we obtain (a).¹³ \square

We are ready to prove Theorem D. It is a corollary of the following theorem.

Theorem 5.13. *Let A be an abelian variety over \overline{K} and let X be an irreducible closed subvariety of A . Suppose that $\dim X/G_X \geq \text{nd-rk } A/G_X$. Then there exists a special point $x_0 \in X(\overline{K})$ such that $X = x_0 + G_X$ if X has dense small points.¹⁴*

Proof. Let $\phi : A/G_X \rightarrow A'_* \times B'$ be an isogeny of abelian varieties, where B' is a representative of the nowhere degenerate factor for A/G_X .

Suppose that X has dense small points. Then so does X/G_X by [29, Lemma 2.1], and hence X/G_X is tropically trivial by Theorem 5.4. Since $\text{nd-rk}(A'_*) = 0$ (cf. Remark 5.8 (1)), it follows from Proposition 5.12 that there exists a point $a' \in A'_*(\overline{K})$ such that $Y := \phi(X) \subset \{a'\} \times B'$. Then we obtain $Y = \{a'\} \times B'$ since $\dim Y \geq \text{nd-rk } A/G_X = \dim B'$ by our assumption. Since X/G_X has trivial stabilizer and ϕ is an isogeny, we find $B' = 0$, which concludes that $Y = \{a'\}$. On the other hand, Y has dense small points by [29, Lemma 2.1], and hence a' is a special point of A'_* as we mentioned just above Conjecture 5.2. By [29, Lemma 2.10], we can then take a special point $x_0 \in X(\overline{K})$ such that $a' = \phi(\theta(x_0))$, where $\theta : A \rightarrow A/G_X$ is the quotient homomorphism. Consequently, we have $X/G_X = \{\theta(x_0)\}$ and hence $X = x_0 + G_X$ as required. \square

¹³This argument also shows that X is tropically trivial if there is an isogeny $\phi : A \rightarrow A' \times B'$ such that B' is nowhere degenerate and that $\phi(X) \subset \{a'\} \times B'$ for some $a' \in A'(\overline{K})$.

¹⁴In the setting of this theorem, note that $\dim X/G_X = 0$ and hence $\text{nd-rk } A/G_X = 0$ as a result.

Corollary 5.14 (Theorem D). *Let A be an abelian variety over \overline{K} with $\text{nd-rk } A \leq 1$. Then the geometric Bogomolov conjecture holds for A .*

Proof. Let X be an irreducible closed subvariety of A . It is enough to show that if X has dense small points, then X/G_X consists of one special point of A/G_X .

Assume that X has dense small points. Suppose that $\dim X/G_X > 0$. Then we have

$$\dim X/G_X \geq 1 \geq \text{nd-rk } A \geq \text{nd-rk } A/G_X,$$

where the last inequality follows from Corollary 5.9. Therefore we find that X/G_X is just a special point by Theorem 5.13, but that contradicts our assumption that $\dim X/G_X > 0$. Accordingly, we have $\dim X/G_X = 0$ in this situation, i.e., X/G_X consists of a single point x' . Since $X/G_X = \{x'\}$ has dense small points, the point x' is a special point, which shows our corollary. \square

5.6. Geometric Bogomolov conjecture for curves. We conclude this section by mentioning the geometric Bogomolov conjecture for curves.¹⁵ it insists that the set of \overline{K} -points of a non-isotrivial smooth projective curve of genus more than 1 should be “discrete” in its Jacobian with respect to the Néron-Tate seminorm.

To be precise, let C be a smooth projective curve over \overline{K} of genus $g \geq 2$. Let J_C be its jacobian variety, D a divisor $C(\overline{K})$ of degree 1 and let $j_D : C \rightarrow J_C$ be the embedding given by $x \mapsto [x - D]$, where $[x - D]$ denote the divisor class of $x - D$. Let $\|\cdot\|_{NT}$ be the canonical Néron-Tate semi-norm arising from the canonical Néron-Tate pairing on J_C . Then the Bogomolov conjecture for curves is the following:

Conjecture 5.15. For any $P \in J_C(\overline{K})$, there should exist an $\epsilon > 0$ such that

$$\{x \in C(\overline{K}) \mid \|j_D(x) - P\|_{NT} < \epsilon\}$$

is a finite set if C is not isotrivial.

Here C is said to be *isotrivial* if there is a curve C' over k such that $C'_K \cong C$. A stronger version of this conjecture is also well known as the effective geometric Bogomolov conjecture for curves:

Conjecture 5.16. Suppose that C is not isotrivial. Then there should exist an $\epsilon > 0$ such that

$$\{x \in C(\overline{K}) \mid \|j_D(x) - P\|_{NT} < \epsilon\}$$

is a finite set for any $P \in J_C(\overline{K})$. Moreover, if C has a stable model over \mathfrak{B} , then we can describe such an ϵ effectively in terms of geometric information of this stable model.

In char $K = 0$, after partial results by Zhang [30], Moriwaki [21, 22, 23], and the author [27, 28], Cinkir proved Conjecture 5.16 in [10]. In positive characteristic, Conjecture 5.15 as well as Conjecture 5.16 are unsolved in full generality, and there are only some partial answers: Conjecture 5.16 is solved when

- the stable model of C has only irreducible fibers ([23]),

¹⁵In this subsection, we assume that \mathfrak{B} is a curve, namely, that K is a function field of one variable, because some of the early results discussed in this subsection needs that assumption. We should note that Theorem 5.17 holds true and the proof here works well even in the case where K is the function field of a higher dimensional variety.

- C is a curve of genus 2 ([21]),
- C is a hyperelliptic curve ([28]), or
- C is non-hyperelliptic and $g = 3$ ([27]).

The geometric Bogomolov conjecture for abelian varieties can imply Conjecture 5.15.¹⁶ To see that, let \hat{h} be the Néron-Tate height such that $\hat{h}(x) = \|x\|_{NT}^2$. Then we have $\hat{h}(j_{D+P}(x)) = \|j_D(x) - P\|_{NT}^2$ for all $P \in J_C(\overline{K})$. Accordingly, Conjecture 5.15 is equivalent to saying that $j_D(C)$ does not have dense small points for any divisor D on C of degree 1, unless C is isotrivial. To show its contraposition, we assume that $j_D(C)$ has dense small points for some D . Then $j_D(C)$ is a special subvariety of J_C since Conjecture 5.2 is supposed to hold true. By the definition of special subvarieties and the fact that the trace homomorphism is purely inseparable (cf. [19, VIII, § 3, Corollary 2] or [29, Lemma 1.4]), there is a smooth projective curve C_0 over k and a purely inseparable finite morphism $\phi : (C_0)_{\overline{K}} \rightarrow C$. Let $C_0 \rightarrow C_0^{(q)}$ be the q -th relative Frobenius morphism, where q is the degree of ϕ . Then its base-change to \overline{K} is also the q -th relative Frobenius morphism, and hence we have $(C_0^{(q)})_{\overline{K}} \cong C$ by [25, Corollary 2.12]. Thus we conclude that C is isotrivial.

The following result is obtained as a corollary of Theorem 5.4, without assumption on the characteristic. Recall here that a curve C over \overline{K} is of *compact type at v* if the special fiber of the stable model of C_v is a tree of smooth irreducible components. It is well known to be equivalent to saying that the jacobian variety of C_v is non-degenerate (cf. [8, Chapter 9]).

Theorem 5.17. *Suppose that there exists a place at which C is of non-compact type. Then Conjecture 5.15 holds true for C .*

Proof. If C is of non-compact type at v , the jacobian J_C is degenerate at v . We can therefore take a simple abelian variety A' over \overline{K} degenerate at v and a surjective homomorphism $\phi : J_C \rightarrow A'$.

Let D be any divisor on C of degree 1. Since J_C itself is the smallest abelian subvariety of J_C containing $j_D(C) - x_0$, where $x_0 \in j_D(C(\overline{K}))$, the image of $j_D(C) - x_0$ by ϕ cannot be a point. Therefore, we find that $j_D(C)$ is tropically non-trivial by Lemma 5.11. Since $j_D(C)$ has trivial stabilizer, we conclude that $j_D(C)$ cannot have dense small points by Theorem 5.4, which implies Conjecture 5.15 for C as mentioned above. \square

Here we give a couple of remarks. In characteristic zero, Conjecture 5.15 can be deduced from the combination of [23, Theorem E] and Theorem 5.17. Therefore we can avoid hard analysis on metric graphs carried out in [10], as far as we consider the non-effective version only. If the inequality of [23, Theorem D] can be show also in positive characteristic, then the same proof as that of [23, Theorem E] works, and hence we can obtain Conjecture 5.15. Thus our argument makes some contribution to Conjecture 5.15. However, we should also note that our approach does not say anything on the effective version Conjecture 5.16.

6. REDUCTION TO THE CONJECTURE FOR NOWHERE DEGENERATE ABELIAN VARIETIES

In this section, we show Theorem E, which insists that Conjecture 5.2 should follow from the geometric conjecture for nowhere degenerate abelian varieties.

¹⁶The argument from here to the end of the proof of Theorem 5.17 works well even if \mathfrak{B} is a higher dimensional variety.

6.1. Isogeny and special subvarieties. Let $\phi : A \rightarrow B$ be an isogeny of abelian varieties over \overline{K} and let $X \subset A$ be an irreducible closed subvariety. [29, Lemma 2.3] says that X has dense small points if and only if so does $\phi(X)$. That shows us that, if our formulation of the geometric Bogomolov conjecture is correct, then X being special should be equivalent to $\phi(X)$ being special. In fact, we will establish this assertion:

Proposition 6.1. *Let $\phi : A \rightarrow B$ be an isogeny of abelian varieties over \overline{K} and let $X \subset A$ be an irreducible closed subvariety. Then X is a special subvariety if and only if so is $Y := \phi(X)$.*

Proof. It follows immediately from [29, Proposition 2.11] that the image of a special subvariety is also a special subvariety.

In order to show the other implication, suppose that Y is special. Since we have $\phi(G_X) \subset G_Y$, we may assume G_X and G_Y are trivial by taking the quotient. Taking translation by a torsion point if necessary, we may assume $Y = \text{Tr}_B(Y'_K)$ for a closed variety $Y' \subset B^{\overline{K}/k}$. Recall that, for a homomorphism $\phi : A \rightarrow B$ of abelian varieties over \overline{K} , we have a unique homomorphism $\text{Tr}(\phi) : A^{\overline{K}/k} \rightarrow B^{\overline{K}/k}$ characterized by $\text{Tr}_B \circ \text{Tr}(\phi)_{\overline{K}} = \phi \circ \text{Tr}_A$. Since $\text{Tr}(\phi)$ is surjective by [29, Lemma 1.5], we have $\text{Tr}(\phi)(\text{Tr}(\phi)^{-1}(Y')) = Y'$. Taking account that

$$\text{Tr}(\phi)^{-1}(Y')_{\overline{K}} = (\text{Tr}(\phi)_{\overline{K}})^{-1}(Y'_K)$$

we then see that

$$\phi(\text{Tr}_A(\text{Tr}(\phi)^{-1}(Y')_{\overline{K}})) = \text{Tr}_B(\text{Tr}(\phi)_{\overline{K}}((\text{Tr}(\phi)_{\overline{K}})^{-1}(Y'_K))) = \text{Tr}_B(Y'_K) = Y.$$

Since ϕ is an isogeny and X is irreducible, we can therefore take a torsion point $\tau \in A(\overline{K})$ such that $Z := X - \tau$ is an irreducible component of $\text{Tr}_A(\text{Tr}(\phi)^{-1}(Y')_{\overline{K}})$. Since k is algebraically closed, we can take an irreducible component Z' of $\text{Tr}(\phi)^{-1}(Y')$ with $Z'_K = Z$, and we have $\text{Tr}_A(Z'_K) + \tau = X$. Thus we conclude that X is a special subvariety. \square

Corollary 6.2. *Let $\phi : A \rightarrow B$ a homomorphism of abelian varieties over \overline{K} .*

- (1) *Suppose that ϕ is an isogeny. Then the geometric Bogomolov conjecture holds for A if and only if it holds for B .*
- (2) *Suppose that ϕ is surjective. If the geometric Bogomolov conjecture holds for A , then it holds for B .*

Proof. The assertion (1) immediately follows from [29, Lemma 2.3] and Proposition 6.1. To show (2), we take an abelian subvariety $B' \subset A$ such that $\phi|_{B'} : B' \rightarrow B$ is an isogeny. It is easy to see that the geometric Bogomolov conjecture for A implies that for B' . Accordingly, it follows from (1) that the conjecture also holds for B . \square

6.2. Reduction to the case of nowhere degenerate abelian varieties. Now we can establish one of our main results:

Theorem 6.3 (Theorem E). *Let A be an abelian variety and let B be a representative of the nowhere degenerate factor for A . Then the following are equivalent to each other:*

- (a) *The geometric Bogomolov conjecture holds for A .*
- (b) *The geometric Bogomolov conjecture holds for B .*

Proof. By virtue of Corollary 6.2 (1), we may assume $A = A_* \times B$ for some abelian variety A_* . Then it immediately follows from Corollary 6.2 (2) that (a) implies (b).

Let us prove that (b) implies (a). Suppose that the conjecture holds for B . Let $\theta : A \rightarrow A/G_X$ be the quotient, and take an isogeny $\phi : A/G_X \rightarrow A'_* \times B'$ of abelian varieties, where B' is a representative of the nowhere degenerate factor for A/G_X . Note that $\text{nd-rk}(A'_*) = 0$. Let $p' : A'_* \times B' \rightarrow A'_*$ and $q' : A'_* \times B' \rightarrow B'$ be the canonical projections. Applying Lemma 5.6 to $\phi \circ \theta$, we obtain a homomorphism $\psi' : B \rightarrow B'$ for $\phi \circ \theta$, and it is surjective since $\phi \circ \theta$ is surjective. By Corollary 6.2 (2), we see that the geometric Bogomolov conjecture holds for B' .

Let X be an irreducible closed subvariety of A having dense small points. Then X/G_X is tropically trivial by Theorem 5.4, and hence $Y := \phi(X/G_X)$ is also tropically trivial (cf. (1.3.11)). Therefore, Proposition 5.12 allows us to pick a point $a' \in A'_*(\overline{K})$ such that $Y \subset \{a'\} \times B'$. The point a' is a special point of A'_* since $\{a'\}$ is the image of X by $p' \circ \phi \circ \theta$ and X has dense small points, and hence $(a', 0_{B'}) \in A'_* \times B'$ is a special point. We put $Y' := Y - (a', 0_{B'})$, which is a subvariety of $\{0\} \times B' = B'$. Taking account that X has dense small points, we see that Y also has dense small points and so does Y' . Since the conjecture holds for B' , it follows that Y' is a special subvariety. That implies that $Y = Y' + (a', 0_{B'})$ is special (cf. [29, Remark 2.6]), and hence X/G_X is also special by Proposition 6.1. Accordingly, we conclude that X is special by [29, Proposition 2.11], and thus we obtain our theorem. \square

The following assertion follows immediately since $\text{nd-rk}(A) \leq \dim A$:

Corollary 6.4. *Let s be a non-negative integer. The following are equivalent to each other:*

- (a) *Conjecture 5.2 holds for any abelian variety A with $\text{nd-rk } A \leq s$.*
- (b) *Conjecture 5.2 holds for any nowhere degenerate abelian variety B with $\dim B \leq s$.*

Note that Corollary 5.14 also follows from Corollary 6.4 since the geometric Bogomolov conjecture holds for elliptic curves.

As a result, Corollary 6.4 says in particular that the geometric Bogomolov conjecture in full generality is equivalent to the following conjecture.

Conjecture 6.5 (Geometric Bogomolov conjecture for nowhere degenerate abelian varieties). *Let A be a nowhere degenerate abelian variety over \overline{K} and let X be a closed subvariety of A . Then X should not have dense small points unless it is a special subvariety.*

In the proof of any theorems on the Bogomolov conjecture for abelian varieties, the equidistribution theorem was a crucial ingredient. However, our arguments so far suggest that equidistribution theorem on Berkovich analytic spaces should be useless against the proof of Conjecture 6.5 because the canonical subset, which is the support of the canonical measures, is just a single point in this setting. They do not have enough information to lead us to any result in this case. A different strategy should be constructed for this conjecture.

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